# Multiple integrals

Lecture notes for the students – mathematicians, IV Semester

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Multiple integrals. Lecture notes for the students – mathematicians, IV Semester. Упорядн. D. Leviatan та I. O. Shevchuk. – Електронне видання. 2024. – 39. с.

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Рекомендовано для студентів другого курсу спеціальностей "математика" "статистика".

Рекомендовано до друку Вченою радою механіко-математичного факультету Київського національного університету імені Тараса Шевченка 23 січня 2024 року, протокол N9.

# XIV Multiple integrals

25 лютого 2024 р.

## Зміст

# General notations

- $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

- $\mathbf{x} = (x_1, \dots, x_m)$, $\mathbf{x}_1 = (x_{1,1}, \dots, x_{1,m})$, $\mathbf{x}^* = (x_1^*, \dots, x_m^*)$, $\mathbf{a} = (a_1, \dots, a_m)$
  $\mathbf{e}_1 := (1, 0, \dots, 0)$, $\mathbf{e}_2 := (0, 1, 0, \dots, 0)$, $\dots$, $\mathbf{e}_m := (0, \dots, 0, 1)$,
  $$\mathbf{x}^t := \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}.$$

- The symbol " : " means *"such that"*.

- The letter $G$ will always designate an open set.

- The letter $F$ will always designate a closed set.

- For $E \subset \mathbb{R}^m$, let $\overline{E}$ denote the closure of the set $E$, and $E^0$ its interior. Denote by
  $$\partial E := \overline{E} \setminus E^0.$$
  the *boundary* of the set $E$. In particular,
  $$E^0 = E \setminus \partial E = \overline{E} \setminus \partial E, \qquad \overline{E} = E \cup \partial E = E^0 \cup \partial E.$$
  Remark, each closed interval
  $$[\mathbf{x}', \mathbf{x}''] := \{\mathbf{x} = t\mathbf{x}' + (1-t)\mathbf{x}'' : \ t \in [0,1]\}$$
  with the ends $\mathbf{x}' \in E$ and $\mathbf{x}'' \notin E$ contains a point $\mathbf{x} \in \partial E$.
  Say, $\mathbf{x} = t^*\mathbf{x}' + (1-t^*)\mathbf{x}''$, where $t^* := \sup\{t \in [0,1] : t\mathbf{x}' + (1-t)\mathbf{x}'' \in E\}$.

- Everywhere below all functions $f : E \mapsto \mathbb{R}$ and $g : E \mapsto \mathbb{R}$ are bounded. Write
  $$M(f, E) := \sup_{\mathbf{x} \in E} |f(\mathbf{x})|.$$

# 1 Jordan measure

## 1.1 Elementary Configurations.

**Definition 1.1** (of $n$-cube and its volume). *Let $n \in \mathbb{N}_0$. We call the set $q_n \subset \mathbb{R}^m$ an $n$-cube, if there are $m$ integers $j_1, \ldots, j_m$, such that*

$$q_n = \left\{ \mathbf{x} \in \mathbb{R}^m : \frac{j_1}{2^n} \leq x_1 \leq \frac{j_1 + 1}{2^n}, \ldots, \frac{j_m}{2^n} \leq x_m \leq \frac{j_m + 1}{2^n} \right\}.$$

*The volume of $n$-cube is denoted by $|q_n|$, and we naturally define $|q_n| := \frac{1}{2^{nm}}$.*

**Definition 1.2** (of elementary configuration and its volume). *We call a nonempty set $A \subset \mathbb{R}^m$ elementary configuration, if it is composed of a finite number, say $l \in \mathbb{N}$, of $n$-cubes. Its volume is*

$$|A| := l|q_n| = \frac{l}{2^{nm}}.$$

*Also regard the empty set $\emptyset$ as an elementary configuration and set $|\emptyset| := 0$.*

**Remark 1.3.** *Each $n$-cube $q_n$ is composed of $2^m$ $(n+1)$-cubes $q_{n+1}$. Hence, if an elementary configuration $A$ is composed of $l$ $n$-cubes, then it is also composed of $l_1 = 2^m l$ $(n+1)$-cubes. Note that*

$$|A| = l_1|q_{n+1}| = \frac{l2^m}{2^{m(n+1)}} = \frac{l}{2^{nm}} = l|q_n|,$$

*independent of its decomposition level. Therefore, it is convenient to define the level of an elementary configuration $A$, as the minimal $n$ for which $A$ is composed of $n$-cubes.*

Let the elementary configurations $A$ and $B$ be of levels $n_1$- and $n_2$, respectively. Then each of them is composed of $n$-cubes. where $n = \max\{n_1, n_2\}$. This implies

**Lemma 1.4** (On elementary configurations). *Let $A$ and $B$ be elementary configurations. Then a) their intersection and union are also elementary configurations and*

$$|A \cup B| \leq |A| + |B|;$$

*b) if $A^0$ and $B^0$ are disjoint, that is, if $A^0 \cap B^0 = \emptyset$, then*

$$|A \cup B| = |A| + |B|;$$

*c) if $B \subset A$, then*

$$|B| \leq |A|;$$

*d) the closure of their difference, that is, $\overline{A \setminus B}$, is an elementary configuration, and if $B \subset A$, then*

$$|\overline{A \setminus B}| = |A| - |B|.$$

## 1.2 Definition of the Jordan measure

**Definition 1.5** (of interior and exterior configurations). *An interior configuration of a bounded set $E \subset \mathbb{R}^m$ is a set $E_{(n)}$, composed of all $n$-cubes $q_n$, such that $q_n \subset E$. An exterior configuration of a bounded set $E \subset \mathbb{R}^m$ is a set $E^{(n)}$, composed of all $n$-cubes $q_n$, such that $q_n \cap E \neq \emptyset$.*

**Remark 1.6.** *By definition, $E_{(n)}$ and $E^{(n)}$ are elementary configurations satisfying,*

$$E_{(n)} \subset E_{(n+1)} \subset E^{(n+1)} \subset E^{(n)},$$

*which implies*

$$|E_{(n)}| \leq |E_{(n+1)}| \leq |E^{(n+1)}| \leq |E^{(n)}|.$$

*Hence, both sequences $\left\{|E_{(n)}|\right\}_{n=0}^{\infty}$ and $\left\{|E^{(n)}|\right\}_{n=0}^{\infty}$ are monotone and bounded sequences of nonnegative numbers. Therefore we may state the following definition.*

**Definition 1.7** (of interior and exterior measures). *The interior Jordan measure of a bounded set $E \subset \mathbb{R}^m$ is*

$$\mu_* E := \lim_{n\to\infty} |E_{(n)}| = \sup_{n\in\mathbb{N}} |E_{(n)}|.$$

*The exterior Jordan measure of a bounded set $E \subset \mathbb{R}^m$ is*

$$\mu^* E := \lim_{n\to\infty} |E^{(n)}| = \inf_{n\in\mathbb{N}} |E^{(n)}|.$$

**Remark 1.8.** *Note that the interior and exterior measures are defined for any bounded set $E$. Moreover,*

$$\mu_* E \leq \mu^* E < \infty.$$

*Also, if $H \subset E$, then*

$$H_{(n)} \subset E_{(n)} \quad \text{and} \quad H^{(n)} \subset E^{(n)}, \quad \text{so that} \quad \mu_* H \leq \mu_* E \quad \text{and} \quad \mu^* H \leq \mu^* E.$$

**Definition 1.9** (of Jordan measure). *A set $E \subset \mathbb{R}^m$ is called Jordan measurable (or simply, measurable), if it is bounded and its interior and exterior Jordan measures are equal. Its Jordan measure $\mu E$ is defined as*

$$\mu E := \mu_* E = \mu^* E.$$

**Remark 1.10.** *Note that a bounded set $E$ for which $\mu^* E = 0$ is necessarily measurable and $\mu E = 0$.*

**Example 1.11.** *(Measure in $\mathbb{R}^2$ of the unit square ). Let $Q = [0,1]^2$ – be a unit square. Then, $Q^{(n)} = [-2^{-n}, 1 + 2^{-n}]^2$ for all $n$, and $Q_{(n)} = Q$, whence $\mu^* Q = \mu_* Q = 1 = \mu Q$.*

**Example 1.12.** *(The set $E \subset \mathbb{R}^2$ is not Jordan measurable). Let $\mathbb{Q}$ be the set of rational numbers, and $Q = [0,1]^2$ be the unit square. Set $E := Q \cap \mathbb{Q}^2$. Then $E^{(n)} = [-2^{-n}, 1 + 2^{-n}]^2$ for all $n$ and $E_{(n)} = \emptyset$. Hence $\mu^* E = 1 \neq 0 = \mu_* E$, so that $E$ is not a measurable set.*

## 1.3 Criteria of measurability.

Let $E \subset \mathbb{R}^m$ be a bounded set. Denote by

$$\Delta E_{(n)} := \overline{E^{(n)} \setminus E_{(n)}} = E^{(n)} \setminus (E_{(n)})^0,$$

the closure of the difference of exterior and interior configurations of the set $E$.

113 **Theorem 1.13** ($\Delta$ – criterion of measurability ). *For the set $E$ to be measurable, it is necessary and sufficient, that*

$$\lim_{n\to\infty} |\Delta E_{(n)}| = 0.$$

*Доведення.* By the definition of exterior and interior measures,

$$\mu^* E - \mu_* E = \lim_{n\to\infty} |E^{(n)}| - \lim_{n\to\infty} |E_{(n)}| = \lim_{n\to\infty} \left(|E^{(n)}| - |E_{(n)}|\right) = \lim_{n\to\infty} |\overline{E^{(n)} \setminus E_{(n)}}|$$
$$= \lim_{n\to\infty} |\Delta E_{(n)}|,$$

where the penultimate equality follows from the property d) of elementary configurations.

□

114 **Theorem 1.14** (Approximating criterion of measurability). *For the set $E$ to be measurable, it is necessary and sufficient, that $\forall \epsilon > 0$ there are sets $E_1, E_2 \subset \mathbb{R}^m$, such that*

$$E_1 \subset E \subset E_2 \quad \textit{and} \quad \mu^* E_2 - \mu_* E_1 < \epsilon.$$

*Доведення.* Necessity is trivial, since one may take $E_1 = E_2 := E$. In order to prove sufficiency, we observe that $E_1 \subset E \subset E_2$ implies

$$\mu_* E_1 \le \mu_* E \le \mu^* E \le \mu^* E_2,$$

hence

$$\mu^* E - \mu_* E \le \mu^* E_2 - \mu_* E_1 < \epsilon.$$

Since $\epsilon$ is arbitrary, it follows that $\mu^* E = \mu_* E$. □

115 **Theorem 1.15** (Boundary criterion of measurability). *For the set $E$ to be measurable, it is necessary and sufficient, that its boundary be measurable of the measure zero.*

*Доведення.* It is easy to check the inclusion (see also Lemma 1.17 below),

$$\partial E \subset \Delta E_{(n)} \subset (\partial E)^{(n)},$$

which in turn implies

$$(\partial E)^{(n)} \subset (\Delta E_{(n)})^{(n)}.$$

Since for each $n$-cube $q_n$, $|q_n^{(n)}| = 3^m |q_n|$, we conclude that $|(\Delta E_{(n)})^{(n)}| \le 3^m |\Delta E_{(n)}|$. Hence

$$|\Delta E_{(n)}| \le |(\partial E)^{(n)}| \le 3^m |\Delta E_{(n)}|,$$

and the proof follows by Theorem 1.13. □

## 1.4 On the boundary of a set.

Recall that in the General notations we agreed to denote by $\overline{E}$, the closure of the set $E$, and by $E^0$, the set of interior points of the set $E \subset \mathbb{R}^m$. Finally, we denoted

$$\partial E = \overline{E} \setminus E^0,$$

the boundary of the set $E$. Every point $\mathbf{x} \in \partial E$ is called a *boundary point* of $E$.

**Remark 1.16.** *Recall also, that*
*a) the inclusions* $\quad E^0 \subset E \subset \overline{E} \quad$ *hold;*
*b)* $\overline{F} = F \;\Leftrightarrow\; F \quad$ *is a closed set;*
*c)* $G^0 = G \;\Leftrightarrow\; G \quad$ *is an open set;*
*and*
*d)* $\quad H \subset E \;\Rightarrow\; H^0 \subset E^0 \quad$ *and* $\quad \overline{H} \subset \overline{E}$.

117 **Lemma 1.17** (On the boundary of a set)**.** *If $E$ is a bounded set, then*

$$\partial E \subset \Delta E_{(n)} \subset (\partial E)^{(n)}.$$

*Доведення.* Since $E_{(n)} \subset E \subset E^{(n)}$, we conclude that $(E_{(n)})^0 \subset E^0$ and $\overline{E} \subset \overline{E^{(n)}} = E^{(n)}$. Therefore

$$\partial E = \overline{E} \setminus E^0 \subset \overline{E} \setminus (E_{(n)})^0 \subset E^{(n)} \setminus (E_{(n)})^0 = \overline{E^{(n)} \setminus E_{(n)}} = \Delta E_{(n)},$$

and the left hand inclusion is proved.

In order to prove the right hand inclusion, let $q_n \subset \Delta E_{(n)}$. Since $q_n \subset E^{(n)}$, there exists an $\mathbf{x}' \in q_n : \mathbf{x}' \in E$. Since $q_n \not\subset E_{(n)}$, there exists an $\mathbf{x}'' \in q_n : \mathbf{x}'' \notin E$. Hence the closed interval

$$[\mathbf{x}', \mathbf{x}''] := \{\mathbf{x} = t\mathbf{x}' + (1-t)\mathbf{x}'' : \; t \in [0,1]\},$$

with the ends $\mathbf{x}'$ and $\mathbf{x}''$ contains a point $\mathbf{x} \in \partial E$. Since $[\mathbf{x}', \mathbf{x}''] \subset q_n$, we conclude that $\mathbf{x} \in q_n$, so that $q_n \subset (\partial E)^{(n)}$. □

**Lemma 1.18** (On the boundaries of sets)**.** *For $E, H \subset \mathbb{R}^m$, we have*

$$\partial(E \cup H) \subset \partial E \cup \partial H \qquad \textit{and} \qquad \partial(E \setminus H) \subset \partial E \cup \partial H.$$

*Доведення.* First we observe that $\overline{E \cup H} = \overline{E} \cup \overline{H}$, since $\overline{E \cup H} \subset \overline{\overline{E} \cup \overline{H}} = \overline{E} \cup \overline{H} \subset \overline{E \cup H}$. This together with $E^0 \subset (E \cup H)^0$ and $H^0 \subset (E \cup H)^0$ implies

$$\begin{aligned}\partial(E \cup H) &= \overline{E \cup H} \setminus (E \cup H)^0 = (\overline{E} \cup \overline{H}) \setminus (E \cup H)^0 \\ &= (\overline{E} \setminus (E \cup H)^0) \cup (\overline{H} \setminus (E \cup H)^0) \subset (\overline{E} \setminus E^0) \cup (\overline{H} \setminus H^0) = \partial E \cup \partial H,\end{aligned}$$

which proves the left hand inclusion. As for the right hand inclusion we have,

$$\partial(E \setminus H) = \overline{E \setminus H} \setminus (E \setminus H)^0 \subset \overline{\overline{E} \setminus H^0} \setminus (E^0 \setminus \overline{H})^0 = (\overline{E} \setminus H^0) \setminus (E^0 \setminus \overline{H}) \subset \partial E \cup \partial H,$$

where only the last inclusion requires a proof. To this end, if $\mathbf{x} \in (\overline{E} \setminus H^0) \setminus (E^0 \setminus \overline{H}) \neq \emptyset$, then $\mathbf{x} \in \overline{E}$ and $\mathbf{x} \notin H^0$. Now, if $\mathbf{x} \in \overline{H}$, then $\mathbf{x} \in \partial H$. Otherwise, $\mathbf{x} \notin \overline{H}$, but $\mathbf{x} \notin E^0 \setminus \overline{H}$, thus $\mathbf{x} \notin E^0$, which together with $\mathbf{x} \in \overline{E}$ yields $\mathbf{x} \in \partial E$. □

## 1.5 Properties of measurable sets.

Let $E$ and $H$ be bounded sets in $\mathbb{R}^m$. The equality $(E \cup H)^{(n)} = E^{(n)} \cup H^{(n)}$, which follows from

$$q_n \subset E^{(n)} \cup H^{(n)} \Leftrightarrow (q_n \subset E^{(n)}) \vee (q_n \subset H^{(n)}),$$

implies that $|(E \cup H)^{(n)}| \le |E^{(n)}| + |H^{(n)}|$. Hence

$$(*) \qquad \mu^*(E \cup H) \le \mu^* E + \mu^* H.$$

119 **Theorem 1.19** (On the properties of measurable sets)**.** *If the sets $E$ and $H$ are measurable, then their union $E \cup H$, difference $E \setminus H$ and intersection $E \cap H$, are also measurable sets.*

*Доведення.* Since $E$ and $H$ are measurable, their boundaries are also measurable, $\mu^*(\partial E) = \mu(\partial E) = 0$ and $\mu^*(\partial H) = \mu(\partial H) = 0$. This and the inclusion $\partial(E \cup H) \subset \partial E \cup \partial H$ yield

$$0 \le \mu_*(\partial(E \cup H)) \le \mu^*(\partial(E \cup H)) \le \mu^*(\partial E \cup \partial H) \le \mu^*(\partial E) + \mu^*(\partial H) = 0,$$

that is $\mu_*(\partial(E \cup H)) = \mu^*(\partial(E \cup H)) = 0 = \mu(\partial(E \cup H))$, so the boundary $\partial(E \cup H)$ is a measurable set of measure zero, and therefore the union $E \cup H$ is a measurable set. Similarly, the inclusion $\partial(E \setminus H) \subset \partial E \cup \partial H$ yields the measurability of the difference $E \setminus H$. Finally, the measurability of the intersection $E \cap H$ follows from the equality $E \cap H = E \setminus (E \setminus H)$. □

120 **Corollary 1.20.** *If a set $E \subset \mathbb{R}^m$ is measurable, then so are its interior $E^0 = E \setminus \partial E$ and its closure $\overline{E} = E \cup \partial E$.*

**Corollary 1.21.** *If $\{E_i\}_{i=1}^n$ are measurable, then so is $\cup_{i=1}^n E_i$.*

122 **Theorem 1.22** (On the properties of Jordan measure)**.** *Let $E$ and $H$ be measurable. Then*
*a) (monotonicity) if $E \subset H$ , then*

$$\mu E \le \mu H;$$

*b) (subadditivity)*

$$\mu(E \cup H) \le \mu E + \mu H;$$

*c) (additivity) if their interiors $E^0$ and $H^0$ are disjoint, that is, if $E^0 \cap H^0 = \emptyset$, then*

$$\mu(E \cup H) = \mu E + \mu H.$$

*Доведення.* *a*) If $E \subset H$ , then $\mu^* E \le \mu^* H$, whence $\mu E = \mu^* E \le \mu^* H = \mu H$.
*b*) By Theorem 1.19, the set $E \cup H$ is measurable, and b) follows from (*).
*c*) Since $E^0 \cap H^0 = \emptyset$, it follows that $(E^0)_{(n)} \cap (H^0)_{(n)} = \emptyset$ and $(E^0 \cup H^0)_{(n)} = (E^0)_{(n)} \cup (H^0)_{(n)}$. Therefore, by the additivity of elementary configurations, $|(E^0 \cup H^0)_{(n)}| = |(E^0)_{(n)}| + |(H^0)_{(n)}|$, whence $\mu_*(E^0 \cup H^0) = \mu_* E^0 + \mu_* H^0$. Since by Theorem 1.19 and Corollary 1.20, $E^0$, $H^0$ and $E^0 \cup H^0$ are measurable, it follows that $\mu(E^0 \cup H^0) = \mu E^0 + \mu H^0$. By virtue of Theorem 1.15, Corollary 1.20, and *a*) and *b*), we get

$$\begin{aligned} \mu E + \mu H &\le \mu \overline{E} + \mu \overline{H} = \mu(E^0 \cup \partial E) + \mu(H^0 \cup \partial H) \le \mu E^0 + \mu \partial E + \mu H^0 + \mu \partial H \\ &= \mu E^0 + \mu H^0 = \mu(E^0 \cup H^0) \le \mu(E \cup H) \le \mu E + \mu H. \end{aligned}$$

Hence,

$$\mu(E \cup H) = \mu E + \mu H.$$

□

**Corollary 1.23.** *If the set $E$ is measurable, then $\mu E = \mu E^0 = \mu \overline{E}$.*

124 **Corollary 1.24.** *If $\{E_i\}_{i=1}^n$ are measurable and their interiors are pairwise disjoint, that is, $E_i^0 \cap E_j^0 = \emptyset$ for all $i \neq j$, then $\mu\Big(\cup_{i=1}^n E_i\Big) = \sum_{i=1}^n \mu E_i$.*

**Lemma 1.25** (On the measure of a cube)**.** *Let $\mathbf{a} \in \mathbb{R}^m$. The cube $Q = [a_1, a_1 + h] \times \cdots \times [a_m, a_m + h]$ (with edges of length $h$, parallel to the coordinate axis), is measurable and,*

$$\mu Q = h^m.$$

*Доведення.* Since $|Q^{(n)}| \leq (h + \frac{2}{2^n})^m$ and, for $n$ sufficiently large, $|Q_{(n)}| \geq (h - \frac{2}{2^n})^m$, it readily follows that $\mu^* Q = \mu_* Q = h^m = \mu Q$. □

**Corollary 1.26.** *Every elementary configuration $A \subset \mathbb{R}^m$ is measurable and, $\mu A = |A|$.*

**Definition 1.27** (The translation mapping)**.** *The mapping $\varphi : \mathbb{R}^m \mapsto \mathbb{R}^m$ defined by $\varphi(\mathbf{x}) := \mathbf{x} + \mathbf{a}$, is called a translation of $\mathbf{x}$ by the vector $\mathbf{a}$.*

**Corollary 1.28.** *The image $\varphi(E)$ of a measurable set $E \subset \mathbb{R}^m$ is a measurable set and $\mu(\varphi(E)) = \mu E$.*

## 1.6 Problems.

Prove the following assertions.

**Problem 1.29.** *The point $\mathbf{x}$ is a boundary point of the set $E$, if every neighborhood of the point $\mathbf{x}$ contains at least two points $\mathbf{x}'$ and $\mathbf{x}''$, such that $\mathbf{x}' \in E$ and $\mathbf{x}'' \notin E$.*

**Problem 1.30.** *The point $\mathbf{x}$ is a boundary point of the set $E$, if there exist two sequences $\{\mathbf{x}_j'\}_{j=1}^\infty$ and $\{\mathbf{x}_j''\}_{j=1}^\infty$, which converge to $\mathbf{x}$, and such that $\mathbf{x}_j' \in E$ and $\mathbf{x}_j'' \notin E$, $\quad \forall j \in \mathbb{N}$.*

**Problem 1.31.** *The inequality $\mu_*(E_1 \cup E_2) \leq \mu_* E_1 + \mu_* E_2$, is not always correct.*

**Problem 1.32.** *Each square in $\mathbb{R}^2$ (with edge length $h$, not necessarily parallel to the coordinate axis), is a measurable set and its measure is equal to $h^2$.*

**Problem 1.33.** *If $E \subset \mathbb{R}^2$ is measurable, then the congruent set is also measurable, and their measures are equal.*

**Problem 1.34.** *Each cube in $\mathbb{R}^3$ is measurable and its measure is equal to $h^3$, where $h$ is the length of the edge of the cube.*

**Remark 1.35.** *A generalized statement for $\mathbb{R}^m$ will be proved later, see Theorem 5.10.*

**Problem 1.36.** *If $E_1$ and $E_2$ are disjoint, or if only their boundaries intersect, then $E_1^0 \cap E_2^0 = \emptyset$.*

**Problem 1.37.** *The converse statement is not always correct.*

# 2 Multiple integrals over measurable sets.

Let $E \subset \mathbb{R}^m$ be a measurable set,

$$\text{diam } E = \sup_{\mathbf{x}' \in E, \mathbf{x}'' \in E} \|\mathbf{x}' - \mathbf{x}''\| = \sup_{\mathbf{x}' \in E, \mathbf{x}'' \in E} \sqrt{(x_1' - x_1'')^2 + \cdots + (x_m' - x_m'')^2}$$

- its diameter, and let $f : E \mapsto \mathbb{R}$ be a bounded function.

## 2.1 Partition of the set. Upper and lower Darboux sums.

**Definition 2.1** (of partition)**.** *A finite collection $\lambda = \{E_j\}_{j=1}^l$ of sets $E_j$ is called a partition of the set $E$, if a) the sets $E_j$ are measurable; b) $\cup_{j=1}^l E_j = E$;*
*c) if $i \neq j$, then the interiors $E_i^0$ and $E_j^0$ of the sets $E_i$ and $E_j$ are disjoint, that is, $E_i^0 \cap E_j^0 = \emptyset$.*
*We call*

$$|\lambda| = \max_{j=1,\ldots,l} \text{diam } E_j,$$

*the diameter of the partition $\lambda = \{E_j\}_{j=1}^l$ of the set $E$.*

**Remark 2.2.** *For every measurable set $E \subset \mathbb{R}^m$ there is a partition of arbitrary small diameter. For if we let $\{q_{n,j}\}_{j=1}^l$ be the collection of $n$-cubes, that compose $E^{(n)}$ and take $\lambda := \{E \cap q_{n,j}\}_{j=1}^l$, then*

$$|\lambda| \leq \text{diam}\, q_n = \frac{\sqrt{m}}{2^n}.$$

**Definition 2.3** (of upper and lower Darboux sums.)**.** *The upper (lower) Darboux sum of the function $f$, associated with the partition $\lambda = \{E_j\}_{j=1}^l$ of the set $E$, is the number*

$$U(f, \lambda) := \sum_{j=1}^{l} \left( \sup_{\mathbf{x} \in E_j} f(\mathbf{x}) \right) \mu E_j \qquad \left( L(f, \lambda) := \sum_{j=1}^{l} \left( \inf_{\mathbf{x} \in E_j} f(\mathbf{x}) \right) \mu E_j \right).$$

**Remark 2.4.** *Since*

$$|\sup_{\mathbf{x} \in E_j} f(\mathbf{x})| \leq |\sup_{\mathbf{x} \in E} f(\mathbf{x})| = M(f, E),$$

*it follows by Corollary 1.24, that*

$$|U(f, \lambda)| = |\sum_{j=1}^{l} \left( \sup_{\mathbf{x} \in E_j} f(\mathbf{x}) \right) \mu E_j| \leq M(f, E) \sum_{j=1}^{l} \mu E_j = M(f, E) \mu E,$$

*and similarly $|L(f, \lambda)| \leq M(f, E)\mu E$. Therefore the next definition is legitimate.*

**Definition 2.5** (of upper and lower integrals)**.** *The upper (lower) integral of the function $f$ over the set $E$ is defined by*

$$\overline{\int}_E f(\mathbf{x})\, d\mathbf{x} := \inf_{\lambda} U(f, \lambda) \qquad \left( \underline{\int}_E f(\mathbf{x})\, d\mathbf{x} := \sup_{\lambda} L(f, \lambda) \right),$$

*where the infimum (supremum) is taken over all partitions $\lambda$ of the set $E$.*

## 2.2 Definition of Riemann integral. Subpartitions.

Let $E \subset \mathbb{R}^m$ be measurable and $f : E \mapsto \mathbb{R}$ be a bounded function. We define subpartition and show that

$$(*) \qquad \underline{\int}_E f(\mathbf{x})d\mathbf{x} \leq \overline{\int}_E f(\mathbf{x})d\mathbf{x}.$$

**Definition 2.6** (of subpartition)**.** *The partition $\lambda' = \{E_i'\}_{i=1}^s$ of the set $E$ is called a subpartition of the partition $\lambda = \{E_j\}_{j=1}^l$ of $E$, if for each $i = 1, \dots, s$ there is $j = 1, \dots, l$, such that $E_i' \subset E_j$.*

**Lemma 2.7** (On subpartition)**.** *If $\lambda'$ be a subpartition of the partition $\lambda$ of the set $E$, then*

$$U(f, \lambda') \leq U(f, \lambda) \qquad \textit{and} \qquad L(f, \lambda') \geq L(f, \lambda).$$

*Доведення.* If $E_i' \subset E_j$, then $\sup_{\mathbf{x} \in E_i'} f(\mathbf{x}) \leq \sup_{\mathbf{x} \in E_j} f(\mathbf{x})$. Hence

$$\begin{aligned} U(f, \lambda') &= \sum_{j=i}^{s} \Big( \sup_{\mathbf{x} \in E_i'} f(\mathbf{x}) \Big) \mu E_i' = \sum_{j=1}^{l} \sum_{i: E_i' \subset E_j} \Big( \sup_{\mathbf{x} \in E_i'} f(\mathbf{x}) \Big) \mu E_i' \\ &\leq \sum_{j=1}^{l} \sum_{i: E_i' \subset E_j} \Big( \sup_{\mathbf{x} \in E_j} f(\mathbf{x}) \Big) \mu E_i' = \sum_{j=1}^{l} \Big( \sup_{\mathbf{x} \in E_j} f(\mathbf{x}) \Big) \sum_{i: E_i' \subset E_j} \mu E_i' \\ &= \sum_{j=1}^{l} \Big( \sup_{\mathbf{x} \in E_j} f(\mathbf{x}) \Big) \mu E_j = U(f, \lambda). \end{aligned}$$

This proves the left hand inequality. The proof of the second inequality is similar. □

**Lemma 2.8** (About joint subpartition)**.** *Let $\lambda_1$ and $\lambda_2$ be two partitions of the set $E$. Then there exists a partition $\lambda'$, which simultaneously is a subpartition of both partitions $\lambda_1$ and $\lambda_2$.*

*Доведення.* Let $\lambda_1 = \{E_j\}_{j=1}^l$ ЧіВіЩІВΘ $\lambda_2 = \{H_i\}_{i=1}^s$. Let $\lambda'$ be the collection of all nonempty sets of the form $E_j \cap H_i$, $j = 1, \dots, l$, $i = 1, \dots, s$. Then clearly $\lambda'$ is a subpartition of both $\lambda_1$ and $\lambda_2$. □

**Remark 2.9.** *Two last lemmas imply the inequalities*

$$L(f, \lambda_1) \leq L(f, \lambda') \leq U(f, \lambda') \leq U(f, \lambda_2),$$

*which yield (*).*

**Definition 2.10** (of Riemann integral)**.** *We say, that $f : E \mapsto \mathbb{R}$ is Riemann integrable on $E$, and write $f \in R(E)$, if its upper and lower integrals over the set $E$ are equal. The Riemann integral of the function $f \in R(E)$ is defined as*

$$\int_E f(\mathbf{x})\, d\mathbf{x} := \overline{\int}_E f(\mathbf{x})\, d\mathbf{x} = \underline{\int}_E f(\mathbf{x})\, d\mathbf{x}.$$

**Remark 2.11.** *From now on instead of "Riemann integral we will simply say "integral".*

212 **Problem 2.12.** *Let $E \subset \mathbb{R}^m$ be a measurable set. Prove: a) $\mu E = 0 \Leftrightarrow E^0 = \emptyset$; b) if $\mu E = 0$, then each function $f$ which is bounded on $E$, is integrable on $E$, and the integral of $f$ over $E$ is equal to zero; c) if $\mu E \neq 0$, then there exists a function $f \notin R(E)$, bounded on $E$.*

## 2.3 Criteria of integrability.

Let $E \subset \mathbb{R}^m$ be a measurable set, and $f : E \mapsto \mathbb{R}$ be a bounded function. Denote

$$\overline{I} := \overline{\int_E} f(\mathbf{x})d\mathbf{x}, \qquad \underline{I} := \underline{\int_E} f(\mathbf{x})d\mathbf{x} \quad \text{and, if } I \text{ exists,} \quad I := \int_E f(\mathbf{x})d\mathbf{x}.$$

**Theorem 2.13** (Criterion of integrability 1.)**.** *A function $f$ is integrable on the set $E$ if and only if for each $\epsilon > 0$ there exists a partition $\lambda$ of the set $E$, such that*

$$(*) \qquad U(f, \lambda) - L(f.\lambda) < \epsilon.$$

*Доведення.* $\Leftarrow$ (sufficiency). By the definition of the upper and lower integrals,

$$0 \le \overline{I} - \underline{I} \le U(f, \lambda) - L(f, \lambda) < \epsilon.$$

Since $\epsilon$ is an arbitrary, this shows that $\overline{I} = \underline{I}$, so that $f \in R(E)$.

$\Rightarrow$ (necessity). By the definition of the upper and lower integrals and the properties of the infimum and supremum, there are partitions $\lambda_1$ and $\lambda_2$, of $E$, such that

$$U(f, \lambda_1) - I = U(f, \lambda_1) - \overline{I} < \frac{\epsilon}{2} \quad \text{and} \quad I - L(f, \lambda_2) = \underline{I} - L(f, \lambda_2) < \frac{\epsilon}{2}.$$

Let $\lambda$ be a subpartition of both $\lambda_1$ and $\lambda_2$. Then

$$U(f, \lambda) - L(f, \lambda) \le U(f, \lambda_1) - L(f, \lambda_2) = (U(f, \lambda_1) - I) + (I - L(f, \lambda_2)) < \epsilon.$$

This completes the proof. □

**Definition 2.14** (of oscillation)**.** *The oscillation of $f$ on the set $E$ is defined as*

$$\omega(f, E) := \sup_{\mathbf{x} \in E} f(\mathbf{x}) - \inf_{\mathbf{x} \in E} f(\mathbf{x}) = \sup_{\mathbf{x}' \in E, \mathbf{x}'' \in E} |f(\mathbf{x}') - f(\mathbf{x}'')|.$$

The definition immediately implies the following properties of oscillations:

a) if $H \subset E$, then

$$\omega(f, H) \le \omega(f, E);$$

b) it is always true that

$$0 \le \omega(f, E) \le 2M(f, E);$$

c) for a partition $\lambda = \{E_j\}_{j=1}^{l}$ of $E$, we have

$$U(f, \lambda) - L(f, \lambda) = \sum_{j=1}^{l} \omega(f, E_j) \mu E_j.$$

**Theorem 2.15** (Criterion of integrability 2.)**.** *A function $f$ is integrable on the set $E$ if and only if for any $\epsilon > 0$ there exists $\delta > 0$, such that for every partition $\lambda$ of the set $E$, such that $|\lambda| < \delta$, the inequality (*) holds.*

*Доведення.* $\Leftarrow$. Sufficiency follows from Theorem 2.1.
$\Rightarrow$ (necessity). The function $f$ is by assumption integrable on $E$. Therefore, the criterion of integrability 1 implies the existence of a partition $\tilde{\lambda} = \{H_i\}_{i=1}^s$ of the set $E$, such that

$$U(f, \tilde{\lambda}) - L(f, \tilde{\lambda}) \leq \frac{\epsilon}{2}.$$

Denote $\Gamma := \cup_{i=1}^s \partial H_i$. Since all $H_i$ are measurable, we have $\mu\Gamma = 0$, which implies that

$$|(\Gamma^{(n)})^{(n)}| \leq 3^m |\Gamma^{(n)}| \to 0, \quad n \to \infty.$$

Hence there exists an $n_0$, such that

$$|(\Gamma^{(n_0)})^{(n_0)}| < \frac{\epsilon}{4M(f,E)}.$$

Let $\delta = \frac{1}{2^{n_0+1}}$, and let $\lambda = \{E_j\}_{j=1}^l$ be an arbitrary partition of the set $E$, satisfying $|\lambda| < \delta$, and consider the difference

$$\sigma := U(f, \lambda) - L(f, \lambda) = \sum_{i=1}^l \omega(f, E_j)\mu E_j.$$

Let $\kappa$ be the set of indices $1 \leq j \leq l$, such that if $j \in \kappa$, then there exists $1 \leq i \leq s$, for which $E_j \subset H_i$. Then $\sigma$ may be represented as $\sigma = \sigma_1 + \sigma_2$, where

$$\begin{aligned}
\sigma_1 &= \sum_{j\in\kappa} \omega(f, E_j)\mu E_j = \sum_{i=1}^s \sum_{j:E_j\subset H_i} \omega(f, E_j)\mu E_j \leq \sum_{i=1}^s \sum_{j:E_j\subset H_i} \omega(f, H_i)\mu E_j \\
&= \sum_{i=1}^s \omega(f, H_i) \sum_{j:E_j\in H_i} \mu E_j \leq \sum_{i=1}^s \omega(f, H_i)\mu H_i \\
&= U(f, \tilde{\lambda}) - L(f, \tilde{\lambda}) < \frac{\epsilon}{2},
\end{aligned}$$

and

$$\begin{aligned}
\sigma_2 &= \sum_{j\notin\kappa} \omega(f, E_j)\mu E_j \leq 2M(f, E) \sum_{j\notin\kappa} \mu E_j \ = 2M(f, E)\mu\left(\cup_{j\notin\kappa} E_j\right) \\
&\leq 2M(f, E)\mu\left((\Gamma^{(n_0)})^{(n_0)}\right) < \frac{\epsilon}{2},
\end{aligned}$$

where we used the inequality $\omega(f, E_j) \leq 2M(f, E)$ and the inclusion $\cup_{j=1:j\notin\kappa}^l E_j \subset (\Gamma^{(n_0)})^{(n_0)}$. In order to prove the latter inclusion, we will show that

$$E_j \in (\Gamma^{(n_0)})^{(n_0)}, \qquad j \notin \kappa.$$

Indeed, if $j \notin \kappa$, then there is at least one $1 \leq i \leq s$ such that $E_j \cap H_i \neq \emptyset$ and $E_j \setminus H_i \neq \emptyset$. Hence, there are two points $\mathbf{x}' \in E_j$ and $\mathbf{x}'' \in E_j$, such that $\mathbf{x}' \in H_i$ and $\mathbf{x}'' \notin H_i$. Thus, the interval $[\mathbf{x}', \mathbf{x}'']$ contains a point $\mathbf{x}_* \in \partial H_i$, so that, $\mathbf{x}_* \in \Gamma$. Since $diam E_j < \delta$, we conclude that for all $\mathbf{x} \in E_j$,

$$\|\mathbf{x} - \mathbf{x}_*\| \leq \|\mathbf{x} - \mathbf{x}'\| + \|\mathbf{x}' - \mathbf{x}_*\| \leq \|\mathbf{x} - \mathbf{x}'\| + \|\mathbf{x}' - \mathbf{x}''\| < 2\delta = \frac{1}{2^{n_0}},$$

which proves that $\mathbf{x} \in (\Gamma^{(n_0)})^{(n_0)}$. We have shown that $E_j \in (\Gamma^{(n_0)})^{(n_0)}$, and the proof is complete. $\square$

## 2.4 The integral as a limit of integral sums.

Let $E \subset \mathbb{R}^m$ be a measurable set, and $f : E \to \mathbb{R}$ be a bounded function.

**Definition 2.16** (of the integral sum). *The integral sum of the function $f$ over the partition $\lambda = \{E_j\}_{j=1}^l$ of the set $E$ and a collection $X = \{\mathbf{x}_j\}_{j=1}^l$ of points $\mathbf{x}_j \in E_j$ is defined as,*

$$S(f, \lambda, X) := \sum_{j=1}^{l} f(\mathbf{x}_j)\mu E_j.$$

**Remark 2.17.** *Given a partition $\lambda = \{E_j\}_{j=1}^l$ of the set $E$. The following equalities hold*

$$\sup_X S(f, \lambda, X) = U(f, \lambda) \quad \textit{and} \quad \inf_X S(f, \lambda, X) = L(f, \lambda), \tag{1}$$

*where the supremum and infimum are taken over all collections $X = \{\mathbf{x}_j\}_{j=1}^l$ of points $\mathbf{x}_j \in E_j$. In particular,*

$$U(f, \lambda) \leq S(f, \lambda, X) \leq U(f, \lambda). \tag{2}$$

218 **Definition 2.18** (of the limit of integral sums). *The limit of the integral sums $S(f, \lambda, X)$, as $|\lambda| \to 0$, is $J$, notation,*

$$\lim_{|\lambda| \to 0} S(f, \lambda, X) = J,$$

*if $\forall \epsilon > 0,\ \exists \delta > 0$, such that for every partition $\lambda = \{E_j\}_{j=1}^l$ of the set $E$ with diameter $|\lambda| < \delta$ and for every collection $X = \{\mathbf{x}_j\}_{j=1}^l$ of points $\mathbf{x}_j \in E_j$ the inequality*

$$|S(f, \lambda, X) - J| < \epsilon.$$

*holds. If such $J$ exists, then we say that the integral sums of the function $f$ converge on $E$.*

219 **Theorem 2.19** (Criterion of integrability 3). *A function $f$ is integrable on the set $E$ if and only if its integral sums converge on $E$. Moreover,*

$$\lim_{|\lambda| \to 0} S(f, \lambda, X) = \int_E f(\mathbf{x}) d\mathbf{x}. \tag{3}$$

*Доведення.* $\Rightarrow$ (necessity ). Let $f \in R(E)$. Put $I := \int_E f(\mathbf{x}) d\mathbf{x}$. Take $\epsilon > 0$. By Criterion of integrability 2, $\exists \delta > 0 : \forall \lambda : |\lambda| < \delta$, the inequality $U(f, \lambda) - L(f, \lambda) < \epsilon$ is valid. Therefore, $\forall \lambda = \{E_j\}_{j=1}^l : |\lambda| < \delta$ and $\forall X = \{\mathbf{x}_j\}_{j=1}^l : \mathbf{x}_j \in E_j$, we have

$$S(f, \lambda, X) - I \leq U(f, \lambda) - I \leq U(f, \lambda) - L(f, \lambda) < \epsilon,$$

and similarly $I - S(f, \lambda, X) < \epsilon$. Hence, $|I - S(f, \lambda, X)| < \epsilon$. This proves both the necessity and (3).
$\Leftarrow$ (sufficiency). Take $\epsilon > 0$. By Definition 2.18, $\exists \lambda = \{E_j\}_{j=1}^l : \forall X = \{\mathbf{x}_j\}_{j=1}^l : \mathbf{x}_j \in E_j$, we have $-\frac{\epsilon}{3} < S(f, \lambda, X) - J < \frac{\epsilon}{3}$. Therefore,

$$U(f, \lambda) - L(f, \lambda) = \sup_X (S(f, \lambda, X) - J) + \inf_X (J - S(f, \lambda, X)) \leq \frac{2\epsilon}{3} < \epsilon.$$

Thus, $f \in R(E)$ by Criterion of integrability 1. □

## 2.5 Integrability of continuous functions

220 **Lemma 2.20** (On the integrability of continuous functions)**.** *If a function $f : F \mapsto \mathbb{R}$ is continuous on a closed measurable set $F \subset \mathbb{R}^m$, then it is integrable on $F$.*

*Доведення.* Let $\epsilon > 0$. In view of subsection 2.2, Problem 2.12, we may assume that $\mu F > 0$. Since $F$ is closed and bounded, in $\mathbb{R}^m$, it is compact, and $f$ is uniformly continuous there. Therefore there exists a $\delta > 0$, such that for each pair of points $\mathbf{x}' \in F$ and $\mathbf{x}'' \in F$, such that $\|\mathbf{x}' - \mathbf{x}''\| < \delta$, we have

$$|f(\mathbf{x}') - f(\mathbf{x}'')| < \frac{\epsilon}{2\mu F}.$$

Let $\lambda = \{F_j\}_{j=1}^s$ be an arbitrary partition of the set $F$, with $|\lambda| < \delta$. Since

$$\omega(f, F_j) \leq \frac{\epsilon}{2\mu F},$$

we get

$$U(f, \lambda) - L(f, \lambda) = \sum_{j=1}^{s} \omega(f, F_j)\mu F_j \leq \frac{\epsilon}{2\mu F} \sum_{j=1}^{s} \mu F_j = \frac{\epsilon}{2} < \epsilon.$$

Thus, the function $f$ is integrable on $F$ by Criterion of integrability 1. □

**Theorem 2.21** (On the integrability of continuous functions)**.** *If a function $f : E \mapsto \mathbb{R}$ is continuous and bounded on a measurable set $E \subset \mathbb{R}^m$, then it is integrable on $E$.*

*Доведення.* Let $\epsilon > 0$. Since the set $E$ is measurable, there exists an $n$, such that

$$\mu E - \mu E_{(n)} \leq \frac{\epsilon}{4M(f, E)}.$$

The interior configuration $E_{(n)}$ is a closed set. Therefore, by Lemma2.20 and Criterion of integrability 1, there exists a partition $\lambda^* = \{E_j\}_{j=1}^s$ of the set $E_{(n)}$, such that

$$\sum_{j=1}^{s} \omega(f, E_j)\mu E_j < \frac{\epsilon}{2}.$$

Set $E_{s+1} := E \setminus E_{(n)}$. Then $E_{s+1}$ is measurable and $\lambda := \{E_j\}_{j=1}^{s+1}$ is a partition of the set $E$. Hence,

$$\begin{aligned} U(f, \lambda) - L(f, \lambda) &= \sum_{j=1}^{s+1} \omega(f, E_j)\mu E_j = \sum_{j=1}^{s} \omega(f, E_j)\mu E_j + \omega(f, E_{s+1})\mu E_{s+1} \\ &< \frac{\epsilon}{2} + 2M(f, E)\mu E_{s+1} = \frac{\epsilon}{2} + 2M(f, E)(\mu E - \mu E_{(n)}) < \epsilon. \end{aligned}$$

Thus, by Criterion of integrability 1, $f$ is integrable on $E$. □

**Problem 2.22.** *Give an example of a bounded domain (an open connected set) in $R^m, m > 1$, which is not measurable.*

## 2.6 Properties of Riemann integral.

**Theorem 2.23** (On the properties of the Riemann integral)**.** *Let $E \subset \mathbb{R}^m$ be a measurable set, and a function $f$ be integrable on $E$. Then a) if $a \in \mathbb{R}$ and $b \in \mathbb{R}$ are constants, and $g \in R(E)$, then*

$$\int_E (af(\mathbf{x}) + bg(\mathbf{x}))d\mathbf{x} = a\int_E f(\mathbf{x})d\mathbf{x} + b\int_E g(\mathbf{x})d\mathbf{x};$$

*b) if $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in E$, then*

$$\int_E f(\mathbf{x})d\mathbf{x} \geq 0;$$

*c) the function $|f|$ is also integrable on $E$ and*

$$\left|\int_E f(\mathbf{x})d\mathbf{x}\right| \leq \int_E |f(\mathbf{x})|d\mathbf{x};$$

*d) if $g \in R(E)$, then $fg \in R(E)$.*

*Доведення.* $a$) Denote $I := \int_E f(\mathbf{x})d\mathbf{x}$, $J := \int_E g(\mathbf{x})d\mathbf{x}$ and $c := |a| + |b| + 1$. Take $\epsilon > 0$. Since $f \in R(E)$ and $g \in R(E)$, Theorem 2.19 and Definition 2.18 imply the existence of $\delta > 0$, such that for every partition $\lambda = \{E_j\}_{j=1}^l$ of the set $E$ with the diameter $|\lambda| < \delta$ and for each collection $X = \{\mathbf{x}_j\}_{j=1}^l$ of points $\mathbf{x}_j \in E_j$ the following inequalities hold

$$|S(f,\lambda,X) - I| < \frac{\epsilon}{c} \quad \text{and} \quad |S(g,\lambda,X) - J| < \frac{\epsilon}{c}.$$

Therefore, for every such partition $\lambda$ and each such collection $X$ of points, the equality

$$S(af + bg, \lambda, X) = aS(f,\lambda,X) + bS(g,\lambda,X)$$

yields the following estimate,

$$|S(af + bg,\lambda,X) - aI - bJ| \leq |a||S(f,\lambda,X) - I| + |b||S(g,\lambda,X) - J| < \frac{|a|\epsilon}{c} + \frac{|b|\epsilon}{c} < \epsilon,$$

which proves $a$)
$b$) For every partition $\lambda$, we have the inequalities $I \geq L(f,\lambda) \geq 0,$ , which prove $b$).
$c$) Comparing the oscillations of the functions $|f|$ and $f$ over an arbitrary set $H \subset E$ we readily see that

$$\omega(|f|, H) \leq \omega(f, H).$$

Therefore, for every partition $\lambda$, we have

$$U(|f|,\lambda) - L(|f|,\lambda) \leq U(f,\lambda) - L(f,\lambda).$$

Thus, the integrability of $|f|$ on $E$ follows from Criterion of integrability 1 (or 2). The inequality is an immediate consequence $a), b)$ and the inequalities $-|f(x)| \leq f(x) \leq |f(x)|$.
$d$) Again, for every $H \subset E$, we have

$$\omega(fg, H) \leq M(f,E)\omega(g,H) + M(g,E)\omega(f,H).$$

Thus, the integrability of $fg$ follows from Criterion of integrability 1 (or 2). □

## 2.7 Additivity of the Riemann integral.

Another important property of the Riemann integral is,

**Theorem 2.24** (Additivity of the Riemann integral). *Let $E_1 \subset \mathbb{R}^m$ and $E_2 \subset \mathbb{R}^m$ be two measurable sets with disjoint interiors $E_1^0$ and $E_2^0$ . A function $f : E_1 \cup E_2 \mapsto \mathbb{R}$ is integrable on the union $E_1 \cup E_2$ if and only if it is integrable on each of sets $E_1$ and $E_2$. In this case the equality*

$$\int_{E_1 \cup E_2} f(\mathbf{x}) d\mathbf{x} = \int_{E_1} f(\mathbf{x}) d\mathbf{x} + \int_{E_2} f(\mathbf{x}) d\mathbf{x}. \tag{1}$$

*holds.*

*Доведення.* Note first that the set $E := E_1 \cup E_2$ is measurable as a union of two measurable sets. Let $\lambda_1 = \{E_{j,1}\}_{j=1}^{l_1}$ be a partition of the set $E_1$, and $\lambda_2 = \{E_{j,2}\}_{j=1}^{l_2}$ be a partition of the set $E_2$.Then

$$\lambda = \{E_j\}_{j=1}^{l_1+l_2} := \{E_{1,1}, \ldots, E_{l_1,1}, E_{1,2}, \ldots, E_{l_2,2}\}, \tag{2}$$

is a partition of the set $E$. Moreover,

$$U(f, \lambda) = U(f, \lambda_1) + U(f, \lambda_2) \quad \text{and} \quad L(f, \lambda) = L(f, \lambda_1) + L(f, \lambda_2). \tag{3}$$

$\Leftarrow$ (sufficiency). Assume that $f \in R(E_1)$ and $f \in R(E_2)$ and prove that $f \in R(E)$. Take $\epsilon > 0$. By the criterion of intagrability 1, there is a partition $\lambda_1$ of the set $E_1$, such that $U(f, \lambda_1) - L(f, \lambda_1) < \frac{\epsilon}{2}$. There is also a partition $\lambda_2$ of the set $E_2$, such that $U(f, \lambda_2) - L(f, \lambda_2) < \frac{\epsilon}{2}$. Then, for the partition $\lambda$ of the set $E$, defined by the equality (2), we have

$$\begin{aligned} U(f, \lambda) - L(f, \lambda) &= U(f, \lambda_1) - L(f, \lambda_1) + U(f, \lambda_2) - L(f, \lambda_2) \\ &< \epsilon. \end{aligned}$$

Thus, $f \in R(E)$ by the criterion of integrability 1. Sufficiency is proved.
$\Rightarrow$ (necessity). Assume, that $f \in R(E)$ and prove that $f \in R(E_1)$. Let $\epsilon > 0$. Then by Criterion of integrability 2, there exists $\delta > 0$, such that for each partition $\lambda$ of the set $E$, such that $|\lambda| < \delta$, the inequality $U(f, \lambda) - L(f, \lambda) < \epsilon$ holds. Let $\lambda_1$ be a partition of the set $E_1$, such that $|\lambda_1| < \delta$ and $\lambda_2$ a partition of the set $E_2$, such that $|\lambda_2| < \delta$. Then for the partition $\lambda$ of the set $E$, defined by the equality (2), we have $|\lambda| < \delta$. Hence,

$$U(f, \lambda_1) - L(f, \lambda_1) \le U(f, \lambda_1) - L(f, \lambda_1) + U(f, \lambda_2) - L(f, \lambda_2) = U(f, \lambda) - L(f, \lambda) < \epsilon.$$

Thus, $f \in R(E_1)$ by Criterion of integrability 1. Similarly, $f \in R(E_2)$, and the necessity is proved.

Finally, for the partitions $\lambda$, $\lambda_1$ and $\lambda_2$, defined in either proof of sufficiency or proof of necessity, the fact of the existence of all three integrals

$$I := \int_{E_1 \cup E_2} f(\mathbf{x}) d\mathbf{x}, \quad I_1 := \int_{E_1} f(\mathbf{x}) d\mathbf{x}, \quad I_2 := \int_{E_2} f(\mathbf{x}) d\mathbf{x}$$

and equality (3) imply, that

$$I - I_1 - I_2 \le U(f, \lambda) - L(f, \lambda_1) - L(f, \lambda_2) = U(f, \lambda) - L(f, \lambda) < \varepsilon$$

and

$$I_1 + I_2 - I \le U(f, \lambda_1) + U(f, \lambda_2) - L(f, \lambda) = U(f, \lambda) - L(f, \lambda) < \varepsilon,$$

that is $|I - I_1 - I_2| < \varepsilon$ for arbitrary $\varepsilon > 0$. This yields equality (1). □

# 3 Cylindrical sets.

**Problem 3.1.** *Prove that if $E \subset \mathbb{R}^m$ is a bouded set, then $E \times 0$ is measurable in $\mathbb{R}^{m+1}$ and of measure zero.*

The definition of a Jordan measurable set and of the Jordan measure depends of the dimension $m$ of the euclidian space. Indeed, while a bounded set $E \subset \mathbb{R}^m$ may or may not be measurable in $\mathbb{R}^m$. As we have seen in Problem 3.1 above, each bounded set $E \subset \mathbb{R}^m$ is always measurable in $\mathbb{R}^{m+1} = \mathbb{R}^m \times \mathbb{R}$ and its measure in $\mathbb{R}^{m+1}$ is equal to zero. Thus, in the sequel, if we deal with a single euclidian space, we will use the same notations as in the previous paragraph. However, if we need simultaneously different euclidian spaces, as in this paragraph, then instead of “measurable"and “measure we will write “$m$-measurable"and “$m$-measure or “$(m+1)$-measurable"and “$(m+1)$-measure or “$p$-measurable"and “$p$-measure". Also, for the space $\mathbb{R}^m$ $\mathbf{x}$, $\mu$ and $\lambda$ will mean the same as in the previous paragraph. In order to avoid confusion, we will use, in the space $\mathbb{R}^{m+1}$, the notation $\tilde{\mathbf{x}} \in \mathbb{R}^{m+1}$, and $\tilde{\mu}$ and $\tilde{\lambda}$, for the Jordan measure and the partition of the set $H \subset \mathbb{R}^{m+1}$, respectively. In particular, for the sake of saving writing for $\tilde{\mathbf{x}} = (x_1, \ldots, x_m, x_{m+1})$, we denote $\mathbf{x} := (x_1, \ldots, x_m)$, and write $\tilde{\mathbf{x}} = (\mathbf{x}, x_{m+1})$.

## 3.1 Simple cylindrical sets.

We call a set of the form $E \times [0, L]$, where $E \subset \mathbb{R}^m$ is $m$-measurable and $L = const \geq 0$, a simple cylindrical set. One easily proves (see proof of Lemma 1.25) that,

**Lemma 3.2.** *Let $q_n \subset \mathbb{R}^m$ be an $m$-dimension $n$-cube, and $L \geq 0$. Then the set*

$$H := q_n \times [0, L] \subset \mathbb{R}^{m+1}$$

*is $(m+1)$-measurable and* $\tilde{\mu} H = L\ \mu q_n$.

**Lemma 3.3.** *Let $E \subset \mathbb{R}^m$ be an $m$-measurable set, and $L \geq 0$. Then the set*

$$H := E \times [0, L] \subset \mathbb{R}^{m+1}$$

*is $(m+1)$-measurable and* $\tilde{\mu} H = L\ \mu E$.

*Доведення.* Denote $H_n := E_{(n)} \times [0, L]$ and $H^n := E^{(n)} \times [0, L]$. Then $H_n \subset H \subset H^n$. By Lemma 3.1 and the additivity of the measure, we have

$$\tilde{\mu} H_n = L\ \mu E_{(n)} \quad \text{and} \quad \tilde{\mu} H^n = L \mu E^{(n)}.$$

The set $E$ is $m$-measurable, so Definition 1.5 of a measurable set yields,

$$\tilde{\mu} H^n - \tilde{\mu} H_n = L(\mu E^{(n)} - \mu E_{(n)}) \to 0, \quad n \to \infty.$$

Therefore the set $H$ is $(m+1)$-measurable by the approximating criterion of measurability. Finally, the relations

$$L\ \mu E = L \lim_{n\to\infty} \mu E_{(n)} = \lim_{n\to\infty} \tilde{\mu} H_n \leq \tilde{\mu} H \leq \lim_{n\to\infty} \tilde{\mu} H^n = L \lim_{n\to\infty} \mu E^{(n)} = L\ \mu E$$

imply $\tilde{\mu} H = L\ \mu E$. □

## 3.2 Integration over simple cylindrical sets.

**Lemma 3.4.** *Let $E \subset \mathbb{R}^m$ be an $m$-measurable set, $K = const \geq 0$, and let*

$$H := E \times [-K, K]$$

*be a cylindrical set. If $f : H \mapsto \mathbb{R}$ is integrable on $H$, then*

$$\int_H f(\tilde{\mathbf{x}}) d\tilde{\mathbf{x}} = \int_E \left( \overline{\int_{-K}^{K}} f(\mathbf{x}, x_{m+1}) dx_{m+1} \right) d\mathbf{x} = \int_E \left( \underline{\int_{-K}^{K}} f(\mathbf{x}, x_{m+1}) dx_{m+1} \right) d\mathbf{x}.$$

*In particular, if, in addition, for each $\mathbf{x} \in E$ there exists $\int_{-K}^{K} f(\mathbf{x}, x_{m+1}) dx_{m+1}$, then*

$$\int_H f(\tilde{\mathbf{x}}) d\tilde{\mathbf{x}} = \int_E \left( \int_{-K}^{K} f(\mathbf{x}, x_{m+1}) dx_{m+1} \right) d\mathbf{x}.$$

*Доведення.* Let $\epsilon > 0$. By Criterion of integrability 2, there is a $\delta > 0$, such that for every partition $\tilde{\lambda}$ of the set $H$ with $|\tilde{\lambda}| < \delta$, the inequalities

$$U(f, \tilde{\lambda}) - \epsilon < \int_H f(\tilde{\mathbf{x}}) d\tilde{\mathbf{x}} := I \quad \text{and} \quad L(f, \tilde{\lambda}) + \epsilon > I.$$

are valid. Let $\lambda = \{E_j\}_{j=1}^{l}$ be an arbitrary partition of $E$, such that $|\lambda| < \sqrt{\delta}$ and $\lambda_1 = \{J_i\}_{i=1}^{s}$ be an arbitrary partition of the $[-K, K] \subset \mathbb{R}$ into intervals $J_i$ of length $|J_i| < \sqrt{\delta}$. Then $\tilde{\lambda} = \{J_i \times E_j\}_{i=1}^{s}{}_{j=1}^{l}$ is a partition of the set $H$, and $|\tilde{\lambda}| < \delta$. Therefore

$$\begin{aligned}
\overline{\int_E} \left( \overline{\int_{-K}^{K}} f(\mathbf{x}, x_{m+1}) dx_{m+1} \right) d\mathbf{x} &\leq \sum_{j=1}^{l} \left( \sup_{\mathbf{x} \in E_j} \overline{\int_{-K}^{K}} f(\mathbf{x}, x_{m+1}) dx_{m+1} \right) \mu E_j \\
\leq \sum_{j=1}^{l} \left( \sup_{\mathbf{x} \in E_j} \sum_{i=1}^{s} \sup_{x_{m+1} \in J_i} f(\mathbf{x}, x_{m+1}) |J_i| \right) \mu E_j &\leq \sum_{j=1}^{l} \sum_{i=1}^{s} \left( \sup_{\mathbf{x} \in E_j} \sup_{x_{m+1} \in J_i} f(\mathbf{x}, x_{m+1}) \right) |J_i| \mu E_j \\
\leq \sum_{j=1}^{l} \sum_{i=1}^{s} \left( \sup_{\tilde{\mathbf{x}} \in E_j \times J_i} f(\tilde{\mathbf{x}}) \right) |J_i| \mu E_j &= \sum_{j=1}^{l} \sum_{i=1}^{s} \left( \sup_{\tilde{\mathbf{x}} \in E_j \times J_i} f(\tilde{\mathbf{x}}) \right) \tilde{\mu}(J_i \times E_j) \\
&= U(f, \tilde{\lambda}) < I + \epsilon,
\end{aligned}$$

where the equality in the penultimate line follows from Lemma 3.2. Hence

$$\overline{\int_E} \left( \underline{\int_{-K}^{K}} f(\mathbf{x}, x_{m+1}) dx_{m+1} \right) d\mathbf{x} \leq \overline{\int_E} \left( \overline{\int_{-K}^{K}} f(\mathbf{x}, x_{m+1}) dx_{m+1} \right) d\mathbf{x} < I + \epsilon.$$

Similarly,

$$\underline{\int_E} \left( \overline{\int_{-K}^{K}} f(\mathbf{x}, x_{m+1}) dx_{m+1} \right) d\mathbf{x} \geq \underline{\int_E} \left( \underline{\int_{-K}^{K}} f(\mathbf{x}, x_{m+1}) dx_{m+1} \right) d\mathbf{x} > I - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, the first statement follows. □

## 3.3 Measurability of cylindrical sets.

**Definition 3.5** (of cylindrical set)**.** *Let $u : E \mapsto \mathbb{R}$ and $v : E \mapsto \mathbb{R}$ be two integrable functions on an $m$-measurable set $E \subset \mathbb{R}^m$, satisfying*

$$u(\mathbf{x}) \leq v(\mathbf{x}), \quad \mathbf{x} \in E.$$

*The set*

$$H := \{\tilde{\mathbf{x}} = (\mathbf{x}, x_{m+1}) \in \mathbb{R}^{m+1} : \mathbf{x} \in E \quad \text{ЧiВiЦІВђЧiВiЦІВ}\Theta \quad u(\mathbf{x}) \leq x_{m+1} \leq v(\mathbf{x})\} \quad (\subset \mathbb{R}^{m+1}),$$

*is called a cylindrical set in the direction of the axis $Ox_{m+1}$ with the basis $E$.*

**Remark 3.6.** *The functions $u$ and $v$ are often called, respectively, the lower and upper covers of the set $H$.*

**Theorem 3.7.** *A cylindrical set is an $(m+1)$-measurable set.*

*Доведення.* Assume first that $u(\mathbf{x}) \equiv 0$. Take $\epsilon > 0$. Since $v \in R(E)$, then by Criterion of integrability 1, there exists a partition $\lambda = \{E_j\}_{j=1}^l$ of the set $E$, such that

$$U(v, \lambda) - L(v, \lambda) < \epsilon.$$

Set $m_j = \inf_{\mathbf{x} \in E_j} v(\mathbf{x})$and $M_j = \sup_{\mathbf{x} \in E_j} v(\mathbf{x})$, and let

$$H_1 := \cup_{j=1}^l E_j \times [0, m_j] \qquad \text{and} \qquad H_2 := \cup_{j=1}^l E_j \times [0, M_j].$$

Then, clearly,

$$H_1 \subset H \subset H_2.$$

By Lemma 3.2 and the additivity of the measure,

$$\tilde{\mu} H_1 = \sum_{j=1}^{l} m_j \mu E_j = L(v, \lambda) \quad \text{and} \quad \tilde{\mu} H_2 = \sum_{j=1}^{l} M_j \mu E_j = U(v, \lambda).$$

Hence $\tilde{\mu} H_2 - \tilde{\mu} H_1 < \epsilon$. Therefore, by the approximating criterion of measurability, $H$ is an $(m+1)$-measurable.

In order to prove Theorem in the general case, let $L := \inf_{\mathbf{x} \in E} u(\mathbf{x})$, and denote

$$H_3 := \{\tilde{\mathbf{x}} = (\mathbf{x}, x_{m+1}) \in \mathbb{R}^{m+1} : \mathbf{x} \in E \quad \text{and} \quad 0 \leq x_{m+1} \leq v(\mathbf{x}) - L\}$$

and

$$H_4 := \{\tilde{\mathbf{x}} = (\mathbf{x}, x_{m+1}) \in \mathbb{R}^{m+1} : \mathbf{x} \in E \quad \text{and} \quad 0 \leq x_{m+1} \leq u(\mathbf{x}) - L\}.$$

By the above, both cylindrical sets $H_3$ and $H_4$ are $(m+1)$-measurable, which in turn implies that the cylindrical sets $\tilde{H}_3 := H_3 + \{(0, \dots, 0, L)\}$ and $\tilde{H}_4 := H_4 + \{(0, \dots, 0, L)\}$ are $(m+1)$-measurable. Finally, the $(m+1)$-measurability of the set $H$ follows from the inclusions $\tilde{H}_3 \setminus \tilde{H}_4 \subset H \subset \overline{\tilde{H}_3 \setminus \tilde{H}_4}$. $\square$

## 3.4 Theorem on integrability over cylindrical sets.

**Theorem 3.8.** *Let $E \subset \mathbb{R}^m$ be $m$-measurable, $u : E \mapsto \mathbb{R}$ and $v : E \mapsto \mathbb{R}$ be two integrable on $E$ functions, satisfying $u(\mathbf{x}) \le v(\mathbf{x}), \quad \mathbf{x} \in E$, and*

$$H := \{\tilde{\mathbf{x}} = (\mathbf{x}, x_{m+1}) \in \mathbb{R}^{m+1} : \mathbf{x} \in E \quad \text{and} \quad u(\mathbf{x}) \le x_{m+1} \le v(\mathbf{x})\}.$$

*If $f : H \mapsto \mathbb{R}$ is integrable on $H$, and $\int_{u(\mathbf{x})}^{v(\mathbf{x})} f(\mathbf{x}, x_m)dx_m$ exists, for all $\mathbf{x} \in E$, then*

$$\int_H f(\tilde{\mathbf{x}})d\tilde{\mathbf{x}} = \int_E \left( \int_{u(\mathbf{x})}^{v(\mathbf{x})} f(\mathbf{x}, x_{m+1})dx_{m+1} \right) d\mathbf{x}. \tag{1}$$

*Доведення.* Denote $K := \sup_{\mathbf{x}\in E}(|u(\mathbf{x})| + |v(\mathbf{x})|)$, $H_K := E \times [-K, K]$ and

$$\hat{f}(\tilde{\mathbf{x}}) := \begin{cases} f(\tilde{\mathbf{x}}), & \tilde{\mathbf{x}} \in H, \\ 0, & \tilde{\mathbf{x}} \in H_K \setminus H. \end{cases}$$

The additivity of the integral and Lemma 3.3 imply,

$$\begin{aligned} \int_H f(\tilde{\mathbf{x}})d\tilde{\mathbf{x}} = \int_{H_K} f(\tilde{\mathbf{x}})d\tilde{\mathbf{x}} &= \int_E \left( \int_{-K}^{K} f(\mathbf{x}, x_{m+1})dx_{m+1} \right) d\mathbf{x} \\ &= \int_E \left( \int_{u(\mathbf{x})}^{v(\mathbf{x})} f(\mathbf{x}, x_{m+1})dx_{m+1} \right) d\mathbf{x}. \end{aligned}$$

□

**Corollary 3.9.** *If $H$ is the cylindrical set from Theorem 3.8, and $f : H \mapsto \mathbb{R}$ is continuous and bounded there, then (1) holds.*

**Corollary 3.10.** *For the cylindrical set $H$ from Theorem 3.8, we have*

$$\mu H = \int_E (v(\mathbf{x}) - u(\mathbf{x}))d\mathbf{x}.$$

**Problem 3.11.** *Prove the following Theorem.*

**Theorem 3.12.** *Let $H$ be the cylindrical set from Theorem 3.8. If $f : H \mapsto \mathbb{R}$ is integrable on $H$, then*

$$\begin{aligned} \int_H f(\tilde{\mathbf{x}})d\tilde{\mathbf{x}} &= \int_E \left( \overline{\int_{u(\mathbf{x})}^{v(\mathbf{x})}} f(\mathbf{x}, x_{m+1})dx_{m+1} \right) d\mathbf{x} \\ &= \int_E \left( \underline{\int_{u(\mathbf{x})}^{v(\mathbf{x})}} f(\mathbf{x}, x_{m+1})dx_{m+1} \right) d\mathbf{x} \end{aligned}$$

# 4 Volume and measure of a parallelepiped.

## 4.1 $p$-parallelepiped.

Given a system $\{\mathbf{a}_j\}_{j=1}^p$, $1 \le p \le m$ of vectors $\mathbf{a}_j = (a_{j1}, \dots, a_{jm}) \in \mathbb{R}^m$, and a point $\mathbf{x}_0 = (x_{01}, \dots, x_{0m}) \in \mathbb{R}^m$.

**Definition 4.1** (of $p$-parallelepiped)**.** *A $p$-parallelepiped $P(\mathbf{a}_1, \dots, \mathbf{a}_p; \mathbf{x}_0)$, associated with the system $\{\mathbf{a}_j\}_{j=1}^p$ of vectors $\mathbf{a}_j \in \mathbb{R}^m$ and the point $\mathbf{x}_0 \in \mathbb{R}^m$, is the set*

$$P(\mathbf{a}_1, \dots, \mathbf{a}_p; \mathbf{x}_0) := \left\{ \mathbf{x} \in \mathbb{R}^m : \mathbf{x} = \mathbf{x}_0 + \sum_{j=1}^p t_j \mathbf{a}_j, \quad t_j \in [0,1] \right\}.$$

*A $p$-parallelepiped $P(\mathbf{a}_1, \dots, \mathbf{a}_p; \mathbf{x}_0)$ is called degenerate if the system $\{\mathbf{a}_j\}_{j=1}^p$ of vectors $\mathbf{a}_j$ is linearly dependent, otherwise it is called non-degenerate.*

In the case of the parallelepiped of the highest dimension, that is, a non-degenerate m-parallelepiped in $\mathbb{R}^m$, we may suppress the $m$ and just refer to it as parallelepiped.

**Remark 4.2.** *It is well known (and easy to check) that a system $\{\mathbf{a}_j\}_{j=1}^p$ of vectors $\mathbf{a}_j$ is linearly dependent if and only if its Gramm determinant*

$$\begin{vmatrix} (\mathbf{a}_1, \mathbf{a}_1) & \cdots & (\mathbf{a}_1, \mathbf{a}_p) \\ \vdots & \vdots & \vdots \\ (\mathbf{a}_p, \mathbf{a}_1) & \cdots & (\mathbf{a}_p, \mathbf{a}_p) \end{vmatrix} = 0.$$

Note that 1-parallelepiped $P(\mathbf{a}_1, \mathbf{x}_0)$ is a closed interval with ends at $\mathbf{x}_0$ and $\mathbf{x}_0 + \mathbf{a}_1$, the length of which, $\|\mathbf{a}_1\|$, is its volume, that is

$$|P(\mathbf{a}_1, \mathbf{x}_0)| := \|\mathbf{a}_1\|.$$

For $p > 1$ the volume of the $p$-parallelepiped will be defined by induction as the "product of the volume of the $p-1$-parallelepiped base by a height namely,

**Definition 4.3** (of the volume of a $p$-parallelepiped)**.** *Let $p > 1$. The volume of the $p$-parallelepiped $P(\mathbf{a}_1, \dots, \mathbf{a}_p; \mathbf{x}_0)$ is defined as*

$$|P(\mathbf{a}_1, \dots, \mathbf{a}_p; \mathbf{x}_0)| := \|\mathbf{h}\| \; |P(\mathbf{a}_1, \dots, \mathbf{a}_{p-1}; \mathbf{x}_0)|,$$

*where $\mathbf{h}$ is defined by*

$$\mathbf{h} := \mathbf{a}_p + \sum_{j=1}^{p-1} \alpha_j \mathbf{a}_j,$$

*where the $\alpha_j$'s are the solutions of the system of equations*

$$(\mathbf{h}, \mathbf{a}_j) = 0, \quad j = 1, \dots, p-1.$$

**Remark 4.4.** *If $P(\mathbf{a}_1, \dots, \mathbf{a}_{p-1}; \mathbf{x}_0)$ is non-degenerate, then the vector $\mathbf{h}$ is unique. Otherwise, we anyway have, $|P(\mathbf{a}_1, \dots, \mathbf{a}_p; \mathbf{x}_0)| = P(\mathbf{a}_1, \dots, \mathbf{a}_{p-1}; \mathbf{x}_0) = 0$.*

## 4.2 Volume of $p$-parallelepiped.

**Lemma 4.5** (About the volume of a $p$-parallelepiped)**.** *The volume of $p$-parallelepiped, associated with the system $\{\mathbf{a}_j\}_{j=1}^p$ and the point $\mathbf{x}_0$ is the square root of the Gramm determinant of this system, i.e.,*

$$|P(\mathbf{a}_1,\ldots,\mathbf{a}_p;\mathbf{x}_0)|^2=\begin{vmatrix}(\mathbf{a}_1,\mathbf{a}_1) & \cdots & (\mathbf{a}_1,\mathbf{a}_p)\\ \vdots & \vdots & \vdots\\ (\mathbf{a}_p,\mathbf{a}_1) & \cdots & (\mathbf{a}_p,\mathbf{a}_p)\end{vmatrix}.$$

*Доведення.* Evidently, for $p=1$, $|P(\mathbf{a}_1,\mathbf{x}_0)|^2=\|\mathbf{a}_1\|^2=(\mathbf{a}_1,\mathbf{a}_1)$. Assume by induction, that Lemma 4.5 is valid for $p-1\geq 1$ and prove it for $p$. By the induction assumption, we get

$$\begin{vmatrix}(\mathbf{a}_1,\mathbf{a}_1) & \cdots & (\mathbf{a}_1,\mathbf{a}_p)\\ \vdots & \vdots & \vdots\\ (\mathbf{a}_p,\mathbf{a}_1) & \cdots & (\mathbf{a}_p,\mathbf{a}_p)\end{vmatrix}=\begin{vmatrix}(\mathbf{a}_1,\mathbf{a}_1) & \cdots & (\mathbf{a}_1,\mathbf{a}_{p-1}) & (\mathbf{a}_1,\mathbf{h}-\sum_{j=1}^{p-1}\alpha_j\mathbf{a}_j)\\ \vdots & \vdots & \vdots & \vdots\\ (\mathbf{a}_p,\mathbf{a}_1) & \cdots & (\mathbf{a}_p,\mathbf{a}_{p-1}) & (\mathbf{a}_p,\mathbf{h}-\sum_{j=1}^{p-1}\alpha_j\mathbf{a}_j)\end{vmatrix}$$

$$=\begin{vmatrix}(\mathbf{a}_1,\mathbf{a}_1) & \cdots & (\mathbf{a}_1,\mathbf{a}_{p-1}) & -\sum_{j=1}^{p-1}\alpha_j(\mathbf{a}_1,\mathbf{a}_j)\\ \vdots & \vdots & \vdots & \vdots\\ (\mathbf{a}_{p-1},\mathbf{a}_1) & \cdots & (\mathbf{a}_{p-1},\mathbf{a}_{p-1}) & -\sum_{j=1}^{p-1}\alpha_j(\mathbf{a}_{p-1},\mathbf{a}_j)\\ (\mathbf{a}_p,\mathbf{a}_1) & \cdots & (\mathbf{a}_p,\mathbf{a}_{p-1}) & -\sum_{j=1}^{p-1}\alpha_j(\mathbf{a}_p,\mathbf{a}_j)+(\mathbf{a}_p,\mathbf{h})\end{vmatrix}$$

$$=\begin{vmatrix}(\mathbf{a}_1,\mathbf{a}_1) & \cdots & (\mathbf{a}_1,\mathbf{a}_{p-1}) & 0\\ \vdots & \vdots & \vdots & \vdots\\ (\mathbf{a}_{p-1},\mathbf{a}_1) & \cdots & (\mathbf{a}_{p-1},\mathbf{a}_{p-1}) & 0\\ (\mathbf{a}_p,\mathbf{a}_1) & \cdots & (\mathbf{a}_p,\mathbf{a}_{p-1}) & (\mathbf{a}_p,\mathbf{h})\end{vmatrix}=(\mathbf{a}_p,\mathbf{h})\begin{vmatrix}(\mathbf{a}_1,\mathbf{a}_1) & \cdots & (\mathbf{a}_1,\mathbf{a}_{p-1})\\ \vdots & \vdots & \vdots\\ (\mathbf{a}_{p-1},\mathbf{a}_1) & \cdots & (\mathbf{a}_{p-1},\mathbf{a}_{p-1})\end{vmatrix}$$

$$=(\mathbf{a}_p,\mathbf{h})|P(\mathbf{a}_1,\ldots,\mathbf{a}_{p-1};\mathbf{x}_0)|^2=(\mathbf{h},\mathbf{h})|P(\mathbf{a}_1,\ldots,\mathbf{a}_{p-1};\mathbf{x}_0)|^2=\|\mathbf{h}\|^2|P(\mathbf{a}_1,\ldots,\mathbf{a}_{p-1};\mathbf{x}_0)|^2$$
$$=|P(\mathbf{a}_1,\ldots,\mathbf{a}_p;\mathbf{x}_0)|^2,$$

where we used the fact that $\mathbf{h}$ is orthogonal to $\mathbf{a}_j$, $j=1,\ldots,p-1$ and that $(\mathbf{a}_p,\mathbf{h})=(\mathbf{h},\mathbf{h})$. □

If $\mathbf{x}_0=\mathbf{0}$, we suppress the reference to $\mathbf{0}$ and write

$$P(\mathbf{a}_1,\ldots,\mathbf{a}_p):=P(\mathbf{a}_1,\ldots,\mathbf{a}_p;\mathbf{0}).$$

Clearly, the volume of a $p$-parallelepiped is independent of the point $\mathbf{x}_0$. Thus, we write

$$|P(\mathbf{a}_1,\ldots,\mathbf{a}_p)|:=|P(\mathbf{a}_1,\ldots,\mathbf{a}_p;\mathbf{x}_0)|.$$

## 4.3 Parallelepiped, the volume of the projection

**Lemma 4.6** (about the volume of the parallelepiped)**.** *The volume of the $m$-parallelepiped $P(\mathbf{a}_1,\ldots,\mathbf{a}_m)$ satisfies*

$$|P(\mathbf{a}_1,\ldots,\mathbf{a}_m)|=\left\|\begin{matrix}a_{1,1} & \cdots & a_{1,m}\\ \vdots & \vdots & \vdots\\ a_{m,1} & \cdots & a_{m,m}\end{matrix}\right\|.$$

*Доведення.* Since it is well known that

$$\begin{pmatrix} (\mathbf{a}_1,\mathbf{a}_1) & \cdots & (\mathbf{a}_1,\mathbf{a}_m) \\ \vdots & \vdots & \vdots \\ (\mathbf{a}_m,\mathbf{a}_1) & \cdots & (\mathbf{a}_m,\mathbf{a}_m) \end{pmatrix} = \begin{pmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \vdots & \vdots \\ a_{m,1} & \cdots & a_{m,m} \end{pmatrix} \begin{pmatrix} a_{1,1} & \cdots & a_{m,1} \\ \vdots & \vdots & \vdots \\ a_{1,m} & \cdots & a_{m,m} \end{pmatrix}.$$

The proof follows immediately from Lemma 4.5. □

**Remark 4.7.** *Lemma* 4.6 *implies that the volume of the projection of the p-parallelepiped* $P(\mathbf{a}_1,\dots,\mathbf{a}_p)\subset\mathbb{R}^m$ *on the subspace* $\mathbb{R}^p\subset\mathbb{R}^m$ *with basis* $\{\mathbf{e}_{j_1},\dots,\mathbf{e}_{j_p}\}$ *is equal to the absolute value of the determinant*

$$\begin{vmatrix} a_{1,j_1} & \cdots & a_{1,j_p} \\ \vdots & \vdots & \vdots \\ a_{p,j_1} & \cdots & a_{p,j_p} \end{vmatrix}.$$

**Lemma 4.8** (About the volume of the projection of an $(m-1)$-parallelepiped)**.** *The volume of the projection* $P^*$*, of the* $(m-1)$*-parallelepiped* $P\subset\mathbb{R}^m$ *on the subspace* $\mathbb{R}^{m-1}$*, is equal to the product of the volume of* $|P|$ *by the modulus of the cosine of the angle between the axis* $\mathbf{e}_m$ *and the the vector orthogonal to* $P$*.*

*Доведення.* Let $P=P(\mathbf{a}_1,\dots,\mathbf{a}_{m-1})$ and set $\mathbf{a}_m:=\mathbf{e}_m=(0,\dots,0,1)$. If $\mathbf{h}$ is the height of the parallelepiped $P(\mathbf{a}_1,\dots,\mathbf{a}_m)$ defined by the Definition 3.3 for $p=m-1$, then the modulus of the cosine of the angle between the axis $\mathbf{e}_m$ and the vector orthogonal to $P$ is obtained from the equality

$$|\cos\alpha| = \frac{|(\mathbf{a}_m,\mathbf{h})|}{\|\mathbf{h}\|} = \frac{1}{\|\mathbf{h}\|}\left(\mathbf{h}-\sum_{j=1}^{m-1}\alpha_j\mathbf{a}_j,\mathbf{h}\right) = \frac{(\mathbf{h},\mathbf{h})}{\|\mathbf{h}\|} = \|\mathbf{h}\|.$$

Hence

$$\begin{aligned} |P||\cos\alpha| &= |P(\mathbf{a}_1,\dots,\mathbf{a}_{m-1})|\|\mathbf{h}\| = |P(\mathbf{a}_1,\dots,\mathbf{a}_m)| \\ &= \left|\begin{vmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \vdots & \vdots \\ a_{m,1} & \cdots & a_{m,m} \end{vmatrix}\right| = \left|\begin{vmatrix} a_{1,1} & \cdots & a_{1,m-1} & a_{1,m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m-1,1} & \cdots & a_{m-1,m-1} & a_{m-1,m} \\ 0 & \cdots & 0 & 1 \end{vmatrix}\right| \\ &= \left|\begin{vmatrix} a_{1,1} & \cdots & a_{1,m-1} \\ \vdots & \vdots & \vdots \\ a_{m-1,1} & \cdots & a_{m-1,m-1} \end{vmatrix}\right| = |P^*|. \end{aligned}$$

□

## 4.4 Pythagorean theorem

We have

**Theorem 4.9** (A Pythagorean theorem)**.** *The square of the volume of the p-parallelepiped $P(\mathbf{a}_1,\dots,\mathbf{a}_p)\subset\mathbb{R}^m$ is equal to the sum of the squares of the volumes of all projections on the subspaces of dimension p, that is,*

$$|P(\mathbf{a}_1,\dots,\mathbf{a}_p)|^2 = \sum_{1\le j_1<\dots<j_p\le m} \begin{vmatrix} a_{1,j_1} & \cdots & a_{1,j_p} \\ \vdots & \vdots & \vdots \\ a_{p,j_1} & \cdots & a_{p,j_p} \end{vmatrix}^2 . \tag{1}$$

Note that for $p=1$ and $p=m$, the Pythagorean theorem is evident. For $p=m-1$ it follows from the previous lemmas. So, in particular, the Pythagorean theorem is already proved for all $1\le p\le m\le 3$.

*Доведення.* We prove the theorem by induction on $m$. To this end we assume by induction that (1) is valid for $m-1\ge 2$, and prove it for $m$. We first observe that it follows directly from the definition of the determinant of order $p$, that

$$\begin{aligned}(m-p)|P(\mathbf{a}_1,\dots,\mathbf{a}_p)|^2 &= (m-p)\begin{vmatrix} (\mathbf{a}_1,\mathbf{a}_1) & \cdots & (\mathbf{a}_1,\mathbf{a}_p) \\ \vdots & \vdots & \vdots \\ (\mathbf{a}_p,\mathbf{a}_1) & \cdots & (\mathbf{a}_p,\mathbf{a}_p) \end{vmatrix} \\ &= \sum_{i=1}^m \begin{vmatrix} (\mathbf{a}_1^{(i)},\mathbf{a}_1^{(i)}) & \cdots & (\mathbf{a}_1^{(i)},\mathbf{a}_p^{(i)}) \\ \vdots & \vdots & \vdots \\ (\mathbf{a}_p^{(i)},\mathbf{a}_1^{(i)}) & \cdots & (\mathbf{a}_p^{(i)},\mathbf{a}_p^{(i)}) \end{vmatrix},\end{aligned}$$

where, for all $j=1,\dots,p$, $\mathbf{a}_j^{(1)} := (0,a_{j,2},\dots,a_{j,m})$, $\mathbf{a}_j^{(m)} := (a_{j,1},\dots,a_{j,m-1},0)$, and $\mathbf{a}_j^{(i)} := (a_{j,1},\dots,a_{j,i-1},0,a_{j,i+1},\dots,a_{j,m})$, if $i=2,\dots,m-1$. Hence,

$$\begin{aligned}(m-p)|P(\mathbf{a}_1,\dots,\mathbf{a}_p)|^2 &= \sum_{i=1}^m \begin{vmatrix} (\mathbf{a}_1^{(i)},\mathbf{a}_1^{(i)}) & \cdots & (\mathbf{a}_1^{(i)},\mathbf{a}_p^{(i)}) \\ \vdots & \vdots & \vdots \\ (\mathbf{a}_p^{(i)},\mathbf{a}_1^{(i)}) & \cdots & (\mathbf{a}_p^{(i)},\mathbf{a}_p^{(i)}) \end{vmatrix} \\ &= \sum_{i=1}^m \sum_{1\le j_1<\dots<j_p\le m, j_1\ne i,\dots,j_p\ne i} \begin{vmatrix} a_{1,j_1} & \cdots & a_{1,j_p} \\ \vdots & \vdots & \vdots \\ a_{p,j_1} & \cdots & a_{p,j_p} \end{vmatrix}^2 \\ &= (m-p) \sum_{1\le j_1<\dots<j_p\le m} \begin{vmatrix} a_{1,j_1} & \cdots & a_{1,j_p} \\ \vdots & \vdots & \vdots \\ a_{p,j_1} & \cdots & a_{p,j_p} \end{vmatrix}^2 .\end{aligned}$$

□

# 5 Mappings of measurable sets.

Recall that $G$ denotes an open set and $F$ denote a closed set. If there are elements of spaces of different dimension in one expression, say $\mathbf{x} \in \mathbf{R}^p$ and $\mathbf{y} \in \mathbf{R}^q$, then we will distinguish the norms in the different spaces by writing $\|\mathbf{x}\|_p$, respectively, $\|\mathbf{y}\|_q$.

## 5.1 Homeomorphisms.

Let $\varphi : E \mapsto \mathbb{R}^m$ be a continuous mapping of the set $E \subset \mathbb{R}^m$ into $\mathbb{R}^m$, and $\varphi(E)$ its image. Recall that a mapping $\varphi$ is called a homeomorphism between $E$ and $\varphi(E)$, if it is a one to one mapping (bijection) between $E$ and $\varphi(E)$ and the inverse mapping $\varphi^{-1}$ is continuous on $\varphi(E)$.

If $\varphi$ is a homeomorphism between the open sets $G \subset \mathbf{R}^m$ and $\varphi(G) \subset \mathbf{R}^m$, then by well known properties of continuous functions, the image of every open or closed subset of $G$ is, respectively, an open or closed subset of $\varphi(G)$. Conversely, every open and closed subset of $\varphi(G)$ is the image, respectively, of an open or closed subset $G$.

**Lemma 5.1** (about the image of a boundary of a closed set)**.** *Let $\varphi$ be a homeomorphism between the open sets $G \subset \mathbf{R}^m$ and $\varphi(G) \subset \mathbf{R}^m$. Then the image of the boundary of $F \subset G$ is the boundary of its image.*

*Доведення.* We have to show that $\varphi(\partial F) = \varphi(F \setminus F^0) = \varphi(F) \setminus (\varphi(F))^0 = \partial(\varphi(F))$. To that end, first since $F^0$ the interior of $F$ is an open set, it follows by the above that $\varphi(F^0) \subset \varphi(F)$ is open. Hence, $\varphi(F^0) \subset (\varphi(F))^0$. Conversely, we apply the last inclusion to $\varphi(F)$ acted upon by the homeomorphism $\varphi^{-1}$. Namely, $\varphi^{-1}((\varphi(F))^0) \subset (\varphi^{-1}(\varphi(F)))^0 = F^0$, which in turn implies $(\varphi(F))^0 \subset \varphi(F^0)$. Thus, we conclude that $\varphi(F^0) = (\varphi(F))^0$. This completes the proof. □

## 5.2 Continuously differentiable mappings.

Let $G \subset \mathbb{R}^p$ be an open set. A mapping

$$\varphi(\mathbf{x}) = \begin{pmatrix} \varphi_1(\mathbf{x}) \\ \vdots \\ \varphi_q(\mathbf{x}) \end{pmatrix} \quad : G \mapsto \mathbb{R}^q$$

is called continuously differentiable in $G$ (we write $\varphi \in C^1(G)$ ), if all functions $\varphi_j,\ j = 1, \dots, q$, are continuously differentiable in $G$. We denote by

$$\varphi'(\mathbf{x}) := \begin{pmatrix} \frac{\partial \varphi_1(\mathbf{x})}{\partial x_1} & \cdots & \frac{\partial \varphi_1(\mathbf{x})}{\partial x_p} \\ \vdots & \vdots & \vdots \\ \frac{\partial \varphi_q(\mathbf{x})}{\partial x_1} & \cdots & \frac{\partial \varphi_q(\mathbf{x})}{\partial x_p} \end{pmatrix},$$

the derivative of $\varphi$ at the point $\mathbf{x} \in G$.

**Remark 5.2.** *One may consider the numerical matrix* $D = \{d_{i,j}\}_{i=1}^{p}\,{}_{j=1}^{q}$ *as a point in* $\mathbf{R}^{pq}$*, and it is natural to define its norm by*

$$\|D\|_{pq} := \|(d_{1,1}, \ldots, d_{1,q}, d_{2,1}, \ldots, d_{p,q})\|_{pq} = \sqrt{\sum_{i=1}^{p}\sum_{j=1}^{q} d_{i,j}^2}.$$

*Therefore, it is natural to define the norm of the derivative* $\varphi'(\mathbf{x})$ *of the continuously differentiable mapping* $\varphi$ *at the point* $\mathbf{x} \in G$ *as the norm of the matrix* $\varphi'(\mathbf{x})$*, that is,* $\|\varphi'(\mathbf{x})\|_{pq}$*. Note that* $\|\varphi'\|_{pq}$ *is a continuous in* $G$ *function.*

This definition of the norm is justified in view of the following Lemma (see 12.1.Theorem 2 ).

**Lemma 5.3** (About continuously differentiable mapping)**.** *Let* $G \subset \mathbb{R}^p$ *be an open set and* $\varphi : G \mapsto \mathbb{R}^q$ *be continuously differentiable in* $G$*. If* $[\mathbf{x}', \mathbf{x}''] \subset G$*, then*

$$\|\varphi(\mathbf{x}') - \varphi(\mathbf{x}'')\|_q \le \|\mathbf{x}' - \mathbf{x}''\|_p \max_{\mathbf{x}\in[\mathbf{x}',\mathbf{x}'']} \|\varphi'(\mathbf{x})\|_{pq}.$$

**Lemma 5.4** (about the image of closed sets of the measure zero)**.** *Let* $G \subset \mathbb{R}^m$ *be and open set and* $\varphi : G \mapsto \mathbb{R}^m$ *be continuously differentiable in* $G$*. The image* $\varphi(F)$ *of* $F \subset G$ *of measure zero is a set of measure zero.*

*Доведення.* Since $F \subset G$ is closed, there exists an exterior configuration $F^{(n_0)}$ of $F$, which is contained in $G$. The function $\|\varphi'\|_{m^2} : \mathbf{R}^m \mapsto \mathbf{R}^{2m}$ is continuous on the closed and bounded set $F^{(n_0)}$, so that $\max_{\mathbf{x}\in F^{(n_0)}} \|\varphi'(\mathbf{x})\|_{m^2} =: B$, exists. Let $\epsilon > 0$. Since $\mu F = 0$, there exists $n \ge n_0$, such that

$$|F^{(n)}| < \frac{\epsilon}{(2B\sqrt{m})^m}.$$

For an arbitrary $n$-cube $q_n \subset F^{(n)}$, it follows by Lemma 5.2, that

$$\operatorname{diam}(\varphi(q_n)) \le \operatorname{diam}(q_n) \max_{\mathbf{x}\in F^{(n)}} \|\varphi'(\mathbf{x})\|_{m^2} \le \operatorname{diam}(q_n) \max_{\mathbf{x}\in F^{(n_0)}} \|\varphi'(\mathbf{x})\|_{m^2} = B \operatorname{diam}(q_n).$$

Hence, the set $(\varphi(q_n))$ is contained in a cube of measure $(2B\operatorname{diam}(q_n))^m = (2B\sqrt{m})^m|q_n|$, which implies that the set $\varphi(F)$ is contained in a union of cubes, the measure od which does not exceed $(2B\sqrt{m})^m|F^{(n)}| < \epsilon$. Therefore $\mu^*(\varphi(F)) < \epsilon$, and this implies $\mu(\varphi(F)) = 0$. □

An immediate consequence of the above lemmas and the approximating criterion of measurability is

**Lemma 5.5** (About the measurability of the image of closed measurable sets)**.** *Let* $G \subset \mathbb{R}^m$ *be an open set and* $\varphi : G \mapsto \mathbb{R}^m$ *be a continuously differentiable homeomorphism between* $G$ *and* $\varphi(G)$*. Then the image* $\varphi(F)$ *of a closed measurable set* $F \subset G$ *is a closed measurable set.*

## 5.3 Linear mapping

**Lemma 5.6** (On the linear mapping of a cube)**.** *Let $Q := [0,1]^n \subset \mathbb{R}^m$ be the unit cube*

$$D := \begin{pmatrix} a_{11}, & \dots & a_{m1} \\ \vdots & \vdots & \vdots \\ a_{1m}, & \dots & a_{mm} \end{pmatrix}$$

*be a number matrix, and $L : \mathbb{R}^m \mapsto \mathbb{R}^m$ be the linear mapping, given by the equation*

$$L(\mathbf{x}) = D\mathbf{x}^t.$$

*Then the image $L(Q)$ of the cube $Q$ is a parallelepiped $P(\mathbf{a}_1, \dots, \mathbf{a}_m)$, that is,*

$$P(\mathbf{a}_1, \dots, \mathbf{a}_m) = L(Q).$$

*Доведення.* Let $\mathbf{x} \in Q$, so that there are numbers $t_1 \in [0,1], \dots, t_m \in [0,1]$, such that

$$\mathbf{x} = t_1\mathbf{e}_1 + \dots + t_m\mathbf{e}_m.$$

Since $L(\mathbf{e}_j) = \mathbf{a}_j$, $j = 1, \dots, m$, and $L$ is a liner mapping, we have

$$L(\mathbf{x}) = t_1\mathbf{a}_1^t + \dots + t_m\mathbf{a}_m^t.$$

Therefore $L(\mathbf{x}) \in P(\mathbf{a}_1, \dots, \mathbf{a}_m)$. Similarly, if $\mathbf{x} \notin Q$, then also $L(\mathbf{x}) \notin P(\mathbf{a}_1, \dots, \mathbf{a}_m)$. □

**Corollary 5.7.** *If $\varphi(\mathbf{x}) := L(\mathbf{x}) + \mathbf{x}_0^t$, then the image of a cube $Q$ is a parallelepiped $P(\mathbf{a}_1, \dots, \mathbf{a}_m; \mathbf{x}_0)$.*

**Remark 5.8.** *If $\varphi(\mathbf{x}) := L(\mathbf{x}) + \mathbf{x}_0^t = D\mathbf{x}^t + \mathbf{x}_0^t$, then the following identity holds*

$$\varphi'(\mathbf{x}) \equiv D.$$

**Lemma 5.9** ([On the measurability of a parallelepiped)**.** *A parallelepiped is a measurable set.*

*Доведення.* Since the parallelepiped $P(\mathbf{a}_1, \dots, \mathbf{a}_m, \mathbf{x}_0)$ is an image of the parallelepiped $P := P(\mathbf{a}_1, \dots, \mathbf{a}_m) = P(\mathbf{a}_1, \dots, \mathbf{a}_m, \mathbf{0})$ translated by the vector $\mathbf{x}_0$, it is sufficient to prove the lemma for the parallelepiped $P$. By the previous lemma, $P$ is an image of a cube at linear mapping $L$. Since $L'(\mathbf{x}) \equiv D$, the mapping $L$ is continuously differentiable in $\mathbb{R}^m$. Besides, if the determinant $\det D \neq 0$, then $L$ is a homeomorphism between $\mathbb{R}^m$ Bi $\mathbb{R}^m$, therefore the statement of the lemma follows from from Lemma 5.4 About the measurability of the image of closed measurable sets. If $\det D = 0$, it is easy to check, that $P$ is a set of measure zero, see Problem 2.12. □

## 5.4 Measure of the parallelepiped.

5.2 **Theorem 5.10** (about the measure of a parallelepiped)**.** *The measure of a parallelepiped is equal to its volume.*

We divide the proof into three lemmas. Let $D$ be a square matrix $m \times m$. The linear mapping $L(\mathbf{x}) = D\mathbf{x}$ we call *proper*, if $\mu(L(Q_h)) = |L(Q_h)|$ for each cube $Q_h = [0, h]^m$. Note that if $L$ is a proper mapping, then the equality $\mu(L(q_n)) = |L(q_n)|$ holds also for each $n$-cube $q_n \in \mathbf{R}^m$; this follows from Corollary 1.28 about the translation mapping. First we consider matrices of form

$$D = \begin{pmatrix} 1 & & & & & & \\ 0 & \ddots & & & 0 & & \\ & & 1 & & & & \\ c_{1,j} & \cdots & c_{j-1,j} & c_{j,j} & c_{j+1,j} & \cdots & c_{m,j} \\ & & & & 1 & & \\ & & 0 & & & \ddots & 0 \\ & & & & & & 1 \end{pmatrix}$$

which we call *special*, and the mapping $L(\mathbf{x}) = D\mathbf{x}$, *special.*

**Lemma 5.11.** *Every special mapping is a proper one.*

*Доведення.* We prove the lemma for the unit cube $Q = [0, 1]^m$, the proof for the cube $Q_h$, $h > 0$ is similar. The image $L(Q)$ of the cube $Q$ is a parallelepiped $P := P(\mathbf{a}_1, \ldots, \mathbf{a}_m)$, where $\mathbf{a}_j = L(\mathbf{e}_j) = c_{j,j}\mathbf{e}_j$, and if $i \neq j$, then $\mathbf{a}_i = L(\mathbf{e}_i) = \mathbf{e}_i + c_{i,j}\mathbf{e}_j$. Hence $P$ is a cylindrical set in the direction of the axis $\mathbf{e}_j$ with the basis $P^* = P(\mathbf{e}_1, \ldots, \mathbf{e}_{j-1}, \mathbf{e}_{j+1}, \ldots, \mathbf{e}_m)$ and, by Corollary 3.10 from the theorem about the cylindrical set,

$$\mu P = \int_{P^*} (v(\mathbf{x}) - u(\mathbf{x}))d\mathbf{x} = \int_{P^*} |c_{j,j}| d\mathbf{x} = |c_{j,j}|.$$

On the other hand, by Lemma 4.2, $|P| = |P(\mathbf{a}_1, \ldots, \mathbf{a}_m)| = |D| = |c_{j,j}|$. □

**Lemma 5.12.** *Let $P$ be a parallelepiped such that $\mu(P) = |P|$. If $L$ is a proper mapping, then*

$$\mu(L(P)) = |L(P)|.$$

*Доведення.* Take $\epsilon > 0$. By the definition of the measure, there is a number $n$, such that

$$\mu P^{(n)} - \mu P < \epsilon \quad \text{and} \quad \mu P - \mu P_{(n)} < \epsilon.$$

Since $L$ is a proper mapping, by virtue of Theorem 1.22 c), we have

$$\mu(L(P^{(n)})) = |L(P^{(n)})| = |D||P^{(n)}| \quad \text{and} \quad \mu(L(P_{(n)})) = |L(P_{(n)})| = |D||P_{(n)}|,$$

where $|D| := |\det D|$. Taking into account the inclusion $L(P^{(n)}) \subset L(P) \subset L(P_{(n)})$, we get

$$\begin{aligned} |L(P)| - \epsilon|D| &= |D|(|P| - \epsilon) < |D||P_{(n)}| = \mu(L(P_{(n)})) \\ &\le \mu_*(L(P)) \le \mu^*(L(P)) \le \mu(L(P^{(n)})) = |D||P^{(n)}| < |D|(|P| + \epsilon) \\ &= |L(P)| + \epsilon|D|. \end{aligned}$$

Hence $\mu(L(P)) = |L(P)|$. □

**Lemma 5.13.** *Each nonsingular matrix $D$ is a product of $m$ special matrices.*

*Доведення.* One proves the lemma by the induction over $j$ by means of the equality: if $c_{j,j} \neq 0$, then

$$\begin{pmatrix} 1 & & & & & \\ 0 & \ddots & & & 0 & \\ & & 1 & & & \\ c_{1,j} & \cdots & c_{j-1,j} & c_{j,j} & \cdots & c_{m,j} \\ c_{1,j+1} & \cdots & c_{j-1,j+1} & c_{j,j+1} & \cdots & c_{m,j+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{1,m} & \cdots & c_{j-1,m} & c_{j,m} & \cdots & c_{m,m} \end{pmatrix} =$$

$$\begin{pmatrix} 1 & & & & & \\ 0 & \ddots & & & 0 & \\ & & 1 & & & \\ & & & 1 & & \\ d_{1,j+1} & \cdots & d_{j-1,j+1} & d_{j,j+1} & \cdots & d_{m,j+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{1,m} & \cdots & d_{j-1,m} & d_{j,m} & \cdots & d_{m,m} \end{pmatrix} \times \begin{pmatrix} 1 & & & & & & \\ 0 & \ddots & & & 0 & & \\ & & 1 & & & & \\ c_{1,j} & \cdots & c_{j-1,j} & c_{j,j} & c_{j+1,j} & \cdots & c_{m,j} \\ & & & & 1 & & \\ & & 0 & & & \ddots & 0 \\ & & & & & & 1 \end{pmatrix}$$

where $d_{j,k} = \frac{c_{j,k}}{c_{j,j}}$ for all $k > j$ and $d_{i,k} = c_{i,k} - c_{i,j} d_{j,k}$, if $i \neq j$ and $k > j$. □

*Доведення.* of Theorem (5.10). Let $P$ be a nondegenerate parallelepiped and $L(\mathbf{x}) = D\mathbf{x}$ is a linear mapping such that $P = L(Q)$, where $Q = [0,1]^m$. Using Lemma 5.13,, we represent the matrix $D$ in the form of the product of special matrices $D_i$, i.e. in the form $D = D_1 \times \cdots \times D_m$ and denote $L_i(\mathbf{x}) := D_i(\mathbf{x})$. Since for each parallelepiped $P_* \in \mathbf{R}^m$ and for every $i = 1, \ldots, m$ we have $\mu(L_i(P_*)) = |P_*|$, it follows that $\mu(L(P)) = |P|$. For a nondegenerate parallelepiped, the theorem is proved. For of a degenerate parallelepiped $P$ it is easy to check the equality $\mu P = 0 = |P|$.

□

## 5.5 Modulus of continuity of the mapping.

**Definition 5.14** (of Modulus of continuity of the mapping)**.** *The modulus of continuity of the mapping* $\varphi : F \mapsto \mathbf{R}^q$*, continuous on a closed bounded set* $F \subset \mathbf{R}^p$*, is the function*

$$\omega(t, \varphi, F) := \sup_{\mathbf{x}', \mathbf{x}'' \in F : \|\mathbf{x}' - \mathbf{x}''\|_p \le t} \|\varphi(\mathbf{x}') - \varphi(\mathbf{x}'')\|_q, \quad t \ge 0.$$

**Lemma 5.15** (About properties of the modulus of continuity of the mapping.)**.** *The modulus of continuity of the mapping* $\varphi$ *on a closed bounded set* $F$ *has the properties:*
1) $\omega(0, \varphi, F) = 0$ *and* $\omega(t, \varphi, F) \ge 0, \quad t \ge 0$;
2) $\|\varphi(\mathbf{x}') - \varphi(\mathbf{x}'')\|_q \le \omega(\|\mathbf{x}' - \mathbf{x}''\|_p, \varphi, F), \quad \mathbf{x}', \mathbf{x}'' \in F$;
3) $\omega(t_1, \varphi, F) \le \omega(t_2, \varphi, F)$, *if* $0 \le t_1 \le t_2$;
4) $\omega(t, \varphi, F_1) \le \omega(t, \varphi, F)$, *if* $F_1 \subset F, \ t \ge 0$;
5) $\omega(t, \varphi, F) \le 2 \max_{\mathbf{x} \in F} \|\varphi(\mathbf{x})\|_q, \quad t \ge 0$;
6) $\lim_{t \to 0+} \omega(t, \varphi, F) = 0$.

*Доведення.* Properties 1) – 5) readily follow from the definition of modulus of continuity , and 6) follows from the uniform continuity of the mapping on the compact set $F$. □

**Example 5.16.** *If* $\varphi : F \mapsto \mathbf{R}$*, that is* $\varphi$ *is an "ordinary"function, then* $\omega(diamF, \varphi, F) = \omega(\varphi, F)$*, where* $\omega(\varphi, F)$ *is the oscillation of* $\varphi$ *on* $F$*.*

Let now $G \subset \mathbb{R}^p$ be an open set, and assume that the mapping $\varphi : G \mapsto \mathbb{R}^q$ is continuously differentiable in $G$ and $F \subset G$. Then the modulus of continuity $\omega(\cdot, \varphi', F)$ of the derivative $\varphi'$ on $F$ has also the properies 1) – 6), in which, clearly, one must replace $\varphi$ by $\varphi'$.

**Example 5.17.** *If* $\varphi : \mathbf{R}^\mathbf{p} \mapsto \mathbf{R}^\mathbf{q}$ *is a linear mapping, then* $\omega(t, \varphi', F) \equiv 0$.

**Lemma 5.18.** *Let the mapping* $\varphi$ *be continuously differentiable in* $G$*. If the points* $\mathbf{x}_0$ *and* $\mathbf{x}^0$ *belong to* $F$ *together with the interval* $[\mathbf{x}_0, \mathbf{x}^0]$*, connecting them, and* $\xi \in [\mathbf{x}_0, \mathbf{x}^0]$*, then*

$$\|\varphi(\mathbf{x}^0) - \varphi(\mathbf{x}_0) - \varphi'(\xi)(\mathbf{x}^0 - \mathbf{x}_0)^t\|_q \le \|\mathbf{x}^0 - \mathbf{x}_0\|_p \ \omega(\|\mathbf{x}^0 - \mathbf{x}_0\|_p, \varphi', F).$$

*Доведення.* Denote $\psi(\mathbf{x}) := \varphi(\mathbf{x}) - \varphi(\mathbf{x}_0) - \varphi'(\xi)(\mathbf{x} - \mathbf{x}_0)^t$. Then $\psi(\mathbf{x}_0) = \bar{0}$ and $\psi'(\mathbf{x}) = \varphi'(\mathbf{x}) - \varphi'(\xi), \ \mathbf{x} \in F \subset G$. Note that $\mathbf{x}' = I$ (the identity matrix). Therefore Lemma 5.3 together with the properties 2) and 3) imply

$$\|\psi(\mathbf{x}^0)\|_q = \|\psi(\mathbf{x}^0) - \psi(\mathbf{x}_0)\|_q \le \|\mathbf{x}^0 - \mathbf{x}_0\|_p \sup_{\mathbf{x} \in [\mathbf{x}_0, \mathbf{x}^0]} \|\psi'(\mathbf{x})\|_{pq}$$

$$= \|\mathbf{x}^0 - \mathbf{x}_0\|_p \sup\nolimits_{\mathbf{x} \in [\mathbf{x}_0, \mathbf{x}^0]} \|\varphi'(\mathbf{x}) - \varphi'(\xi)\|_{pq} \le \|\mathbf{x}^0 - \mathbf{x}_0\|_p \ \omega(\|\mathbf{x}^0 - \mathbf{x}_0\|_p, \varphi'.F). \quad \square$$

## 5.6 Regular and admissible mappings

**Definition 5.19** (of admissible mappings). *Let $G \subset \mathbf{R}^m$ be an open set. The mapping $\varphi : G \mapsto \mathbf{R}^m$ is called admissible, if $\varphi : G \mapsto \mathbb{R}^m$ is continuously differentiable in $G$, the image $\varphi(G)$ is an open set, and $\varphi$ is a homeomorphism between $G$ and $\varphi(G)$.*

**Remark 5.20.** *One can prove that if $G \subset \mathbf{R}^m$ is a domain and the mapping $\varphi : G \mapsto \mathbf{R}^m$ is admissible, then its jacobian $\det \varphi'(\mathbf{x})$ does not change the sign in $G$, that is, either $\det \varphi'(\mathbf{x}) \geq 0 \quad \forall \mathbf{x} \in G$, or $\det \varphi'(\mathbf{x}) \leq 0 \quad \forall \mathbf{x} \in G$.*

**Remark 5.21.** *Recall that if the mapping $\varphi : G \mapsto \mathbf{R}^m$ is admissible, then the image $\varphi(F)$ of a closed measurable set $F \subset G$ is also a closed measurable set.*

**Remark 5.22.** *In this part, "IV. Multiple integrals we will use admissible mappings. Regular mappings will be used in the next part "V. Stocks Theorem and surface integrals".*

**Definition 5.23** (of regular mapping). *Let $G \subset \mathbf{R}^m$ be an open set. The mapping $\varphi : G \mapsto \mathbf{R}^m$ is called regular, if it is admissible and its jacobian*

$$\det \varphi'(\mathbf{x}) \neq 0, \quad \forall \mathbf{x} \in G.$$

## 5.7 Measure of the image of a cube

**Lemma 5.24** (about the measure of the image of a cube). *Let $\varphi : G \mapsto \mathbf{R}^m$ be an admissible mapping of $G \subset \mathbb{R}^m$ and let $Q \subset G$ be a cube with the edge length $h$. If $\mathbf{x}_0 \in Q$, then*

$$|\mu(\varphi(Q)) - |\det \varphi'(\mathbf{x}_0)|\mu Q| \leq c(m) K^{m-1} \mu Q \ \omega\left(\sqrt{m}h, \varphi', Q\right),$$

*where $c(m)$ is a constant depending only on $m$, and $K := \max_{\mathbf{x}\in Q} \|\varphi'(\mathbf{x})\|_{m^2}$.*

*Доведення.* Denote

$$\varphi_0(\mathbf{x}) := \varphi(\mathbf{x}_0) + \varphi'(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)^t \quad \text{and} \quad \delta := \max_{\mathbf{x}\in Q} \|\varphi(\mathbf{x}) - \varphi_0(\mathbf{x})\|.$$

By Lemma 5.18,

$$\begin{aligned} \delta =& \max_{\mathbf{x}\in Q} \|\varphi(\mathbf{x}) - \varphi_0(\mathbf{x})\| \leq \max_{\mathbf{x}\in Q} \|\mathbf{x} - \mathbf{x}_0\| \omega(\|\mathbf{x} - \mathbf{x}_0\|, \varphi', Q) \\ \leq& \sqrt{m}h \ \omega(\sqrt{m}h, \varphi', Q). \end{aligned}$$

is

$$\delta \leq 2\|\mathbf{x} - \mathbf{x}_0\| \max_{\mathbf{x}\in Q} \|\varphi'(\mathbf{x})\|_{m^2}.$$

Now denote by $E_\delta$, the set of points $\mathbf{y} \in \mathbb{R}^m$, of distance that does not exceed $2\delta$ from the boundary $\partial(\varphi_0(Q))$ of (the parallelepiped) $\varphi_0(Q)$, that is,

$$E_\delta := \{\mathbf{y} \in \mathbb{R}^m : \ dist(\mathbf{y}, \partial(\varphi_0(Q))) \leq 2\delta\}.$$

We will show that both

$$\varphi_0(Q) \subset \varphi(Q) \cup E_\delta \quad \text{and} \quad \varphi(Q) \subset \varphi_0(Q) \cup E_\delta.$$

We begin with the left inclusion. Let $\mathbf{y}_0 \in \varphi_0(Q)$. We may assume that $\mathbf{y}_0 \notin \varphi(Q)$ for otherwise the inclusion is evident. Let $\mathbf{x} \in Q$ be such that $\mathbf{y}_0 = \varphi_0(\mathbf{x})$, and set $\mathbf{y}_1 := \varphi(\mathbf{x})$. Since $\mathbf{y}_0 \notin \varphi(Q)$ and $\mathbf{y}_1 \in \varphi(Q)$, we conclude that in $[\mathbf{y}_0, \mathbf{y}_1]$, $\exists \mathbf{y}_2 \in \partial\varphi(Q)$. Moreover

$$\|\mathbf{y}_2 - \mathbf{y}_0\| \le \|\mathbf{y}_1 - \mathbf{y}_0\| = \|\varphi(\mathbf{x}) - \varphi_0(\mathbf{x})\| \le \delta.$$

The point $\mathbf{y}_3 := \varphi_0\left(\varphi^{-1}(\mathbf{y}_2)\right)$ satisfies

$$\|\mathbf{y}_2 - \mathbf{y}_3\| = \|\varphi\left(\varphi^{-1}(\mathbf{y}_2)\right) - \varphi_0\left(\varphi^{-1}(\mathbf{y}_2)\right)\| \le \max_{\mathbf{x}\in Q} \|\varphi(\mathbf{x}) - \varphi_0(\mathbf{x})\| = \delta.$$

By Lemma 5.1 $\mathbf{y}_3 \in \partial\varphi_0(Q)$. Hence

$$\begin{aligned} dist(\mathbf{y}_0, \partial(\varphi_0(Q))) &\le \|\mathbf{y}_0 - \mathbf{y}_3\| \le \|\mathbf{y}_0 - \mathbf{y}_2\| + \|\mathbf{y}_2 - \mathbf{y}_3\| \\ &\le 2\delta, \end{aligned}$$

so that $\mathbf{y}_0 \in E_\delta$. The left inclusion is proved, and the proof of the right inclusion is similar.

By Lemma 5.5, the image $\varphi(Q)$ is a measurable set and so is the parallelepiped $\varphi_0(Q)$. Moreover, $\mu\left(\varphi_0(Q)\right) = |\det\varphi'(\mathbf{x}_0)|\mu Q$. The above inclusions imply the inequality

$$|\mu(\varphi(Q)) - |\det\varphi'(\mathbf{x}_0)|\mu Q|| = |\mu(\varphi(Q) - \mu(\varphi_0(Q))| \le \mu^*(E_\delta).$$

Thus, it remains to estimate $\mu^*(E_\delta)$. By Lemma 5.3, the length of the edges of the parallelepiped $\varphi_0(Q)$ do not exceed $Kh$. Let $\Gamma$ be one of $2m$ faces of the parallelepiped $\varphi_0(Q)$, and $\Gamma_\delta := \{\mathbf{y} \in \mathbb{R}^m : \ dist(\mathbf{y}, \Gamma) \le 2\delta\}$. It is easy to see, that $\Gamma_\delta$ contains in some rectangular parallelepiped $P$, the length of one of its edges $4\delta$ and the length of any other edge is bounded by $Kh + 4\delta$. Hence, $\mu^*\Gamma_\delta \le \mu P = 4\delta(Kh + 4\delta)^{m-1}$. Thus,

$$\begin{aligned} \mu^*(E_\delta) &\le c_1(m)(Kh + \delta)^{m-1}\delta \\ &\le c_1(m)(Kh + \sqrt{m}h\ \omega(\sqrt{m}h, \varphi', Q))^{m-1}\sqrt{m}h\omega(\sqrt{m}h, \varphi', Q) \\ &\le c_1(m)(Kh + 2\sqrt{m}Kh)^{m-1}\sqrt{m}h\omega(\sqrt{m}h, \varphi', Q) \\ &= c(m)K^{m-1}h^m\omega(\sqrt{m}h, \varphi', Q), \end{aligned}$$

where $c(m) = c_1(m)(1 + 2\sqrt{m}))^{m-1}\sqrt{m}$. Lemma 5.24 is proved. $\square$

# 6 Change of variables.

## 6.1 Main formulations.

100 **Theorem 6.1** (About the change of variables in a multiple integral). *Let $G \subset \mathbb{R}^m$ and assume that $\varphi : G \mapsto \mathbb{R}^m$ is an admissible mapping. Let $F \subset G$ be a closed measurable set. If $f$ is integrable on $\varphi(F)$, then*

$$\int_{\varphi(F)} f(\mathbf{y})d\mathbf{y} = \int_F f(\varphi(\mathbf{x}))|\det \varphi'(\mathbf{x})|d\mathbf{x}.$$

*In particular, the integral on the right hand side exists.*

**Remark 6.2.** *The case $m = 1$ and $F = [a, b]$, is well known. In this case, of course, $\varphi$ is a function of one variable, continuously differentiable in $G := (c, d) \supset [a, b]$, and image $\varphi(F)$ is a closed interval. Since $\varphi$ is a homeomorphism, either $\varphi$ is nondecreasing on $[a, b]$, or $\varphi$ is nonincreasing there. In the former case we have $\varphi'(x) \geq 0, \quad x \in [a, b]$, and $\varphi(F) = [\varphi(a), \varphi(b)]$. Hence, by the formula for a change of variable in the Riemann integral, we get*

$$\int_{\varphi(F)} f(\mathbf{y})d\mathbf{y} = \int_{\varphi(a)}^{\varphi(b)} f(\mathbf{y})d\mathbf{y} = \int_a^b f(\varphi(\mathbf{x}))\varphi'(\mathbf{x})d\mathbf{x} = \int_F f(\varphi(\mathbf{x}))|\varphi'(\mathbf{x})|d\mathbf{x}.$$

*In the latter case we have $\varphi'(x) \leq 0, \quad x \in [a, b]$, and $\varphi(F) = [\varphi(b), \varphi(a)]$. Therefore,*

$$\begin{aligned}\int_{\varphi(F)} f(\mathbf{y})d\mathbf{y} &= \int_{\varphi(b)}^{\varphi(a)} f(\mathbf{y})d\mathbf{y} = -\int_{\varphi(a)}^{\varphi(b)} f(\mathbf{y})d\mathbf{y} = -\int_a^b f(\varphi(\mathbf{x}))\varphi'(\mathbf{x})d\mathbf{x} \\ &= \int_a^b f(\varphi(\mathbf{x}))|\varphi'(\mathbf{x})|d\mathbf{x} = \int_F f(\varphi(\mathbf{x}))|\varphi'(\mathbf{x})|d\mathbf{x}.\end{aligned}$$

## 6.2 Proof of Theorem 6.1

First we prove the theorem for the case when $F$ is an elementary configuration, composed, say, of $n_0$-cubes. Denote $K := \max_{\mathbf{x}\in F} \|\varphi'(\mathbf{x})\|_{m^2}$. Lemma 5.24 and the properties of the modulus of continuity imply that, if $n \geq n_0$ and $q_n \subset F$ is an $n$-cube, then for each point $\mathbf{x} \in q_n$ the inequality

$$|\mu(\varphi(q_n)) - |\det\varphi'(\mathbf{x})|\mu q_n| \leq c(m)K^{m-1}\omega\left(\frac{\sqrt{m}}{2^n}, \varphi', F\right)\mu q_n,$$

holds. For $n \geq n_0$, the elementary configuration $F$ is composed of $l_n$ $n$-cubes $q_{n,1}, \ldots, q_{n,l_n}$ and $\hat{\lambda}_n := \{q_{n,j}\}_{j=1}^{l_n}$ is a partition of $F$. Therefore $\lambda := \{\varphi(q_{n,j})\}_{j=1}^{l_n}$ is a partition of the set $\varphi(F)$ and

$$|\lambda_n| \leq K|\hat{\lambda}_n| = \frac{K\sqrt{m}}{2^n} \to 0, \qquad n \to \infty.$$

Denote $g(\mathbf{x}) := f(\varphi(\mathbf{x}))|\det\varphi'(\mathbf{x})|$. For every collection $\{\mathbf{x}_{n,j}\}_{j=1}^{l_n}$ of points $\mathbf{x}_{n,j} \in q_{n,j}$, and integral sums $S(g, \hat{\lambda}_n, \{\mathbf{x}_{n,j}\}_{j=1}^{l_n})$ and $S(f, \lambda_n, \{\mathbf{y}_{n,j}\}_{j=1}^{l_n})$, where $\mathbf{y}_{n,j} := \varphi(\mathbf{x}_{n,j})$, we have

$$\begin{aligned}
|S(g,\hat{\lambda}_n, \{\mathbf{x}_{n,j}\}_{j=1}^{l_n}) - S(f, \lambda, \{\mathbf{y}_{n,j}\}_{j=1}^{l_n})| &= |\sum_{j=1}^{l_n} g(\mathbf{x}_{n,j})\mu q_{n,j} - \sum_{j=1}^{l_n} f(\mathbf{y}_{n,j})\mu(\varphi(q_{n,j}))| \\
= |\sum_{j=1}^{l_n} f(\mathbf{y}_{n,j})\left(|\det\varphi'(\mathbf{x}_{n,j})|\mu q_{n,j} - \mu(\varphi(q_{n,j}))\right)| &\leq Mc(m)K^{m-1}\omega\left(\frac{\sqrt{m}}{2^n}, \varphi', F\right)\sum_{j=1}^{l_n}\mu q_{n,j} \\
= Mc(m)K^{m-1}\omega\left(\frac{\sqrt{m}}{2^n}, \varphi', F\right)\mu F =: \alpha_n \ \to 0 \qquad n \to \infty,
\end{aligned}$$

where $M := M(f, \varphi(F)) := \sup_{\mathbf{y}\in\varphi(F)} |f(\mathbf{y})|$. In particular,

$$S(g, \hat{\lambda}_n, \{\mathbf{x}_{n,j}\}_{j=1}^{l_n}) \leq S(f, \lambda, \{\mathbf{y}_{n,j}\}_{j=1}^{l_n}) + \alpha_n \leq U(f, \lambda_n) + \alpha_n.$$

Hence

$$U(g, \hat{\lambda}_n) \leq U(f, \lambda_n) + \alpha_n.$$

This implies,

$$\overline{\int_F} g(\mathbf{x})d\mathbf{x} \leq \lim_{n\to\infty} U(g, \hat{\lambda}_n) \leq \lim_{n\to\infty} U(f, \lambda_n) + \alpha_n = \int_{\varphi(F)} f(\mathbf{y})d\mathbf{y},$$

where we have taken into account, that $f \in R(\varphi(F))$ and that $\lambda_n \to 0$ and $\alpha_n \to 0$, as $n \to \infty$. Similarly,

$$\underline{\int_F} g(\mathbf{x})d\mathbf{x} \geq \int_{\varphi(F)} f(\mathbf{y})d\mathbf{y}.$$

The proof for the case for $F$, an elementary configuration, is complete.

To reduce the general case to the previous one, it is sufficient to consider an arbitrary exterior configuration $F^{(\hat{n})}$, where the number $\hat{n}$ is chosen so that $F^{(\hat{n})} \subset G$, and to apply the property of the additivity of the integral, by setting

$$f^*(\mathbf{x}) := \begin{cases} f(\mathbf{x}), & \mathbf{x} \in F \\ 0 & \mathbf{x} \in F^{(\hat{n})} \setminus F. \end{cases}$$

This completes the proof.

## 6.3 More about the change of variables in a multiple integral.

We also have

200 **Theorem 6.3** (about the change of variables in a multiple integral). *Given $G \subset \mathbb{R}^m$, let the admissible mapping $\varphi : G \mapsto \mathbb{R}^m$ be such that its jacobian is a bounded function in $G$. If both sets $G$ and $\varphi(G)$ are measurable, then for each function $f \in R(\varphi(G))$,*

$$\int_G f(\varphi(\mathbf{x}))|\varphi'(\mathbf{x})|d\mathbf{x} = \int_{\varphi(\mathbf{G})} f(\mathbf{y})d\mathbf{y}. \tag{1}$$

*Доведення.* Let $g(\mathbf{x}) := f(\varphi(\mathbf{x}))|\det \varphi'(\mathbf{x})|$, so first we have to prove that $g \in R(G)$. Let $\epsilon > 0$, then since $G$ is measurable, there exists $n$ such that,

$$\mu(G \setminus G_{(n)}) < \frac{\epsilon}{4M(g, G)},$$

where $M(g, G) := \sup_{\mathbf{x}\in G} |g(\mathbf{x})|$. The mapping $\varphi$ is admissible, so that $\varphi(G_{(n)})$ is measurable as an image of the closed measurable set $G_{(n)}$. Since $f \in R(\varphi(G))$, it follows by Theorem 2.6 about the additivity of the integral, that $f \in R(\varphi(G_{(n)}))$. Therefore Theorem 6.1 yields $g \in R(G_{(n)})$. Criterion of integrability 1 implies the existence of a partition $\lambda = \{G_j\}_{j=1}^{s}$ of the set $G_{(n)}$, such that

$$U(g, \lambda) - L(g, \lambda) < \frac{\epsilon}{2}.$$

Denote $G_{s+1} := G \setminus G_{(n)}$. Then $\lambda^* = \{G_j\}_{j=1}^{s+1}$ is a partition of the set $G$ and,

$$U(g, \lambda^*) - L(g, \lambda^*) = U(g, \lambda) - L(g, \lambda) + \omega(g, G_{s+1})\mu G_{s+1} < \frac{\epsilon}{2} + 2M(g, G)\mu G_{s+1} < \epsilon.$$

Thus $g \in R(G)$ by Criterion of integrability 1. Next we have

$$\lim_{n\to\infty} \mu(\varphi(G_{(n)})) = \mu(\varphi(G)). \tag{2}$$

Indeed, let $\epsilon > 0$ be arbitrary and let $E \subset \varphi(G)$ be a closed set such that,

$$\mu(\varphi(G)) - \mu E < \epsilon. \tag{3}$$

Denote by $F$ the inverse image of $E$, that is, $\varphi(F) = E$. Since the mapping $\varphi$ a homeomorphism, $F$ is a closed set. Therefore $\exists N : F \subset G_{(N)}$. Hence, for all $n \geq N$, $F \subset G_{(n)}$, which, in turn, implies $E \subset \varphi(G_{(n)})$. Since $\epsilon$ is arbitrary, this together with (3), implies (2).

Finally, Theorem 6.1 implies that

$$\int_{G_{(n)}} g(\mathbf{x})d\mathbf{x} = \int_{\varphi(G_{(n)})} f(\mathbf{y})d\mathbf{y},$$

for every $n$. Hence,

$$\begin{aligned}|\int_G g(\mathbf{x})d\mathbf{x} - \int_{\varphi(G)} f(\mathbf{y}))d\mathbf{y}| &= |\int_{G\setminus G_{(n)}} g(\mathbf{x})d\mathbf{x} - \int_{\varphi(G\setminus G_{(n)})} f(\mathbf{y})d\mathbf{y}| \\ &\leq M(g, G)\mu(G \setminus G_{(n)}) + M(f, \varphi(G))\mu(\varphi(G \setminus G_{(n)})) \to 0, \quad n \to 0,\end{aligned}$$

where $M(f, \varphi(G)) := \sup_{\mathbf{y}\in\varphi(G)} |f(\mathbf{y})|$. □

## 6.4 Polar, cylindrical and spherical coordinates.

Important examples of the application of Theorem 6.3 on the change of variables are the following.

**Example 6.4** (Polar coordinates)**.** *Let $D \subset \mathbb{R}^2$ be a disk of radius $R$ with the center at the point $\mathbf{0}$, and let the rectangle $P := [0, R] \times [0, 2\pi] \subset \mathbb{R}^2$. If a function $f \in R(D)$, then*

(6.1)
$$\int_D f(x, y)dxdy = \int_P f(r\cos\alpha, r\sin\alpha)rdrd\alpha.$$

*Доведення.* Denote by $G := P^0$, the interior of the rectangle $P$. Consider the mapping $\varphi : G \to \mathbb{R}^2$, given by the equation

$$\varphi(r, \alpha) = \begin{pmatrix} x(r, \alpha) \\ y(r, \alpha) \end{pmatrix} := \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \end{pmatrix}.$$

This mapping is, evidently, continuously differentiable and its jacobian

$$\det\varphi'(r, \alpha) = \begin{vmatrix} \cos\alpha & -r\sin\alpha \\ \sin\alpha & r\cos\alpha \end{vmatrix} = r,$$

is a bounded function. It is easy to check that $\varphi$ is a homeomorphism between $G$ and

$$\varphi(G) = D^0 \setminus \{(x, y) \in \mathbb{R}^2 : x \geq 0, y = 0\}$$

(but not a homeomorphism between $P$ and $D$!). Therefore the conditions of Theorem 6.3 are satisfied, (but, the conditions of Theorem 6.1 are not satisfied for the sets $P$ and $D$!) By Theorem 6.3 we have

$$\begin{aligned} \int_D f(x, y)dxdy &= \left(\int_{\varphi(G)} + \int_{\partial(\varphi(G))}\right) f(x, y)dxdy = \int_{\varphi(G)} f(x, y)dxdy \\ &= \int_G f(r\cos\alpha, r\sin\alpha)rdrd\alpha = \left(\int_G + \int_{\partial G}\right) f(r\cos\alpha, r\sin\alpha)rdrd\alpha \\ &= \int_P f(r\cos\alpha, r\sin\alpha)rdrd\alpha, \end{aligned}$$

where the property of the additivity of an integral is taken into account. □

**Example 6.5** (Cylindrical coordinates.)**.** *Let $D \subset \mathbb{R}^2$ be the disk of radius $R$ with the center at the point $\mathbf{0}$, let the cylinder $C := D \times [0, H] \subset \mathbb{R}^3$, and let the box $P := [0, R] \times [0, 2\pi] \times [0, H] \subset \mathbb{R}^3$. If a function $f \in R(C)$, then*

222 (6.2)
$$\int_C f(x, y, z)dxdydz = \int_P f(r\cos\alpha, r\sin\alpha, t)rdrd\alpha dt.$$

*Доведення.* Denote by $G := P^0$, the interior of $P$. Consider the mapping $\varphi : G \to \mathbb{R}^3$, given by the equation

$$\varphi(r, \alpha, t) = \begin{pmatrix} x(r, \alpha, t) \\ y(r, \alpha, t) \\ z(r, \alpha, t) \end{pmatrix} := \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \\ t \end{pmatrix}.$$

This mapping is, evidently, continuously differentiable and its jacobian

$$\det \varphi'(r,\alpha,t) = \begin{vmatrix} \cos\alpha & -r\sin\alpha & 0 \\ \sin\alpha & r\cos\alpha & 0 \\ 0 & 0 & 1 \end{vmatrix} = r,$$

is a bounded function. It is easy to check that $\varphi$ is a homeomorphism between $G$ and

$$\varphi(G) = C^0 \setminus \{(x,y,z) \in \mathbb{R}^3 : x \geq 0, y = 0, z \in \mathbb{R}\}$$

(but not a homeomorphism between $P$ and $C$!). Therefore the conditions of Theorem 6.3 are satisfied, (but, the conditions of Theorem 6.1 are not satisfied for the sets $P$ and $C$!) Now (6.2) follows from

$$\int_{\varphi(G)} f(x,y,z)dxdydz = \int_G f(r\cos\alpha, r\sin\alpha, t)rdrd\alpha dt,$$

$P = G \cup \partial G$, $C = \varphi(G) \cup \partial(\varphi(G))$, and the property of the additivity of an integral. □

**Example 6.6** (Spherical coordinates)**.** *Let $B \subset \mathbb{R}^3$ be a ball of radius $R$ with the center at zero, and let the box $P := [0,R] \times [0,2\pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \subset \mathbb{R}^3$. If a function $f \in R(B)$, then*

333 (6.3) $$\int_B f(x,y,z)dxdydz = \int_P f(r\cos\alpha\cos\beta, r\sin\alpha\cos\beta, r\sin\beta)r^2\cos\beta drd\alpha d\beta.$$

*Доведення.* Denote by $G := P^0$, the interior of $P$. Consider the mapping $\varphi : G \to \mathbb{R}^3$, given by the equation

$$\varphi(r,\alpha,\beta) = \begin{pmatrix} x(r,\alpha,\beta) \\ y(r,\alpha,\beta) \\ z(r,\alpha,\beta) \end{pmatrix} := \begin{pmatrix} r\cos\alpha\cos\beta \\ r\sin\alpha\cos\beta \\ r\sin\beta \end{pmatrix}.$$

This mapping is, evidently, continuously differentiable, and its jacobian

$$\det \varphi'(r,\alpha,\beta)| = \begin{vmatrix} \cos\alpha\cos\beta & -r\sin\alpha\cos\beta & -r\cos\alpha\sin\beta \\ \sin\alpha\cos\beta & r\cos\alpha\cos\beta & -r\sin\alpha\sin\beta \\ \sin\beta & 0 & r\cos\beta \end{vmatrix} = r^2\cos\beta$$

is a bounded function. It is easy to check, that $\varphi$ is a homeomorphism between $G$ and

$$\varphi(G) = B^0 \setminus \{(x,y,z) \in \mathbb{R}^3 : x \geq 0, y = 0, z \in \mathbb{R}\}$$

(but not a homeomorphism between $P$ and $B$!). Therefore the conditions of Theorem 6.3 are satisfied, (but, the conditions of Theorem 6.1 are not satisfied for the sets $P$ and $B$!) Now (6.3) follows from .

$$\int_{\varphi(G)} f(x,y,z)dxdydz = \int_G f(r\cos\alpha\cos\beta, r\sin\alpha\cos\beta, r\sin\beta)r^2\cos\beta drd\alpha d\beta,$$

$P = G \cup \partial G$, $B = \varphi(G) \cup \partial(\varphi(G))$ and the property of the additivity of an integral. □